Problem 3: Turbulent flow

Exercise 1

A liquid with density ρ and viscosity μ flows in a tube as sketched in figure 1. The Reynolds number is assumed sufficiently high thus the flow can be considered turbulent. Neglecting the entrance and outlet effects the flow can be assumed fully developed. The fluid is considered incompressible thus all the fluid properties are constants.

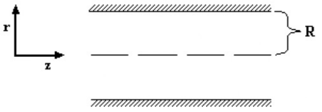


Figure 1: Sketch of a horizontal tube in which we are supposed to introduce a turbulent flow in the direction from left to right.

The radial velocity profile for turbulent flow in a tube can be approximated by (Jakobsen, 2014, page 121):

$$\overline{v_z} \approx v_{max}\left(1-\frac{r}{R}\right)^{1/7} \tag{1}$$

where $\overline{v_z}$ is the time-smoothed velocity.

a) Derive an expression for the cross-sectional average turbulent velocity by using the velocity profile (1). The constant maximum velocity, v_{max}, can be considered known. The averaging operator is:

$$\langle\psi\rangle_A = \frac{1}{A}\iint_A \psi \, da \tag{2}$$

a) $\langle V_z\rangle_A = \frac{1}{A}\iint_A V_z \, da$

Cylinder $\Rightarrow A = \pi R^2$, $da = r\,dr\,d\theta$, $\iint_A = \int_0^{2\pi}\int_0^R$

$$= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R V_{max}\left(1-\frac{r}{R}\right)^{1/7} r\,dr\,d\theta$$

$$= \frac{2V_{max}}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^{1/7} r\,dr$$

, Let $u = 1-\frac{r}{R} \Rightarrow r = R(1-u)$

$\frac{du}{dr} = -\frac{1}{R} \Rightarrow dr = -R\,du$

When $r=0$, $u=1$

$r=R$, $u=0$

$$= \frac{2V_{max}}{\cancel{R^2}}\int_1^0 u^{1/7}\cdot \cancel{R}(1-u)(-\cancel{R})\,du$$

$$= -2V_{max}\int_1^0 u^{1/7} - u^{8/7}\,du$$

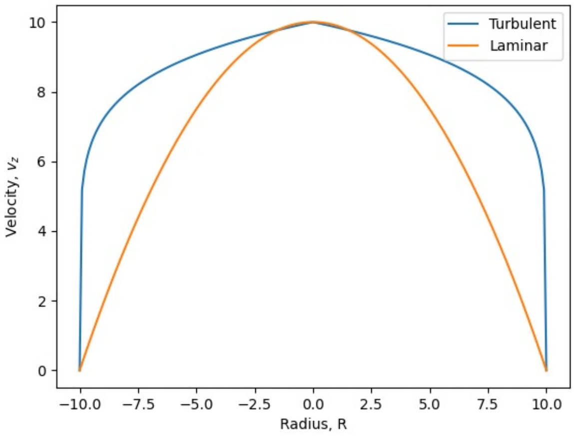
$$= -2V_{max}\left[\frac{7}{8}u^{8/7} - \frac{7}{15}u^{15/7}\right]_1^0$$

$$= -2V_{max}\left[(0) - \left(\frac{7}{8}-\frac{7}{15}\right)\right]$$

$$= 2V_{max}\cdot\frac{49}{120}$$

$$\underline{\underline{\langle V_z\rangle_A = \frac{49}{60}\cdot V_{max}}}$$

b) Sketch the turbulent velocity profile. Include also the laminar profile you did derived in problem 1 in the same plot.




```python
import numpy as np
import matplotlib.pyplot as plt

# Choosing unknown constants
vmax = 10
R = 10  # Radius of pipe

# Creating values to plot
r_values = np.linspace(-R, R, 200)

# The turbulent profile
def v_turbulent(r):
    return vmax*(1-abs(r/R))**(1/7)


# The laminar profile (from exercise 1, problem 1b)
def v_laminar(r):
    return vmax*(1-abs(r/R)**2)


# PLotting the results
plt.plot(r_values, v_turbulent(r_values), label='Turbulent')
plt.plot(r_values, v_laminar(r_values), label='Laminar')
plt.xlabel('Radius, R')
plt.ylabel('Velocity, $v_z$')
plt.legend()
plt.show()
```

The turbulent flow retains its velocity when moving towards the wall better than the laminar flow.

The reason for this is that the inertia forces being larger than the viscous forces in the bulk for the turbulent flow.

Exercise 2

The actual time dependent velocity in a turbulent flow is fluctuating in a chaotic fashion. The fluctuations are irregular deviations around a mean value (see figure 2). The actual time dependent velocity (designated by a tilde) can be written as the sum of a mean velocity (designated with an overbar) and the time dependent fluctuation (designated with a prime). This decomposition is called *Reynolds decomposition*.

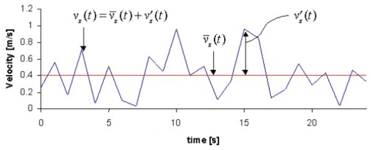


Figure 2: A sketch of a time dependent fluctuating velocity profile around the mean value.

a) Consider the x-component of the Navier-Stokes equation written on the conservative form in Cartesian coordinates. Employ the definitions of the momentum fluxes (stresses) and substitute the fluxes by their definitions in the equation. Moreover, assume that the fluid is incompressible and simplify the equation accordingly. Insert the decomposed velocities and pressure variables.

Notation: Im denoting turbulent variables "normally" ex: V_x, and decomposing into $V_x = \bar{V}_x + V_x'$ where $\bar{V}_x$ is the mean and V_x' is the fluctuation

Starting with the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

Assuming incompressible fluid => constant ρ => $\frac{\partial \rho}{\partial t} = 0$, and $\nabla \cdot (\rho V) = \rho \nabla \cdot V$

$$\Rightarrow \rho \nabla \cdot V = 0$$

$$\underline{\nabla \cdot V = 0}$$

x-component:

$$\frac{\partial}{\partial t}(\rho v_x) + \frac{\partial}{\partial x}(\rho v_x v_x) + \frac{\partial}{\partial y}(\rho v_y v_x) + \frac{\partial}{\partial z}(\rho v_z v_x)$$
$$= -\frac{\partial p}{\partial x} - \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{yx}}{\partial y} - \frac{\partial \sigma_{zx}}{\partial z} + \rho g_x$$

RHS: Inserting the newtonian stress definitions into RHS

$$= -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x}\left(-\mu\left[2\frac{\partial V_x}{\partial x} - \frac{2}{3}\underbrace{(\nabla \cdot V)}_{=0 \text{ from continuity equation}}\right]\right) - \frac{\partial}{\partial y}\left(-\mu\left[\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right]\right) - \frac{\partial}{\partial z}\left(-\mu\left[\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right]\right) + \rho g_x$$

Assuming constant μ

$$= -\frac{\partial p}{\partial x} + \mu\left[2\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_y}{\partial y \partial x} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial^2 V_z}{\partial z \partial x}\right] + \rho g_x$$

Using that the velocity profiles are continous, $\frac{\partial^2 V_y}{\partial y \partial x} = \frac{\partial^2 V_y}{\partial x \partial y}$ and $\frac{\partial^2 V_z}{\partial z \partial x} = \frac{\partial^2 V_z}{\partial x \partial z}$

Then, we can collect terms with $\frac{\partial}{\partial x}$:

$$= -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} + \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)}_{= \nabla \cdot V = 0 \text{ from continuity}}\right] + \rho g_x$$

$$= -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right]+\rho g_x$$

The "full" equation is:

$$\frac{\partial}{\partial t}(\rho V_x)+\frac{\partial}{\partial x}(\rho V_x V_x)+\frac{\partial}{\partial y}(\rho V_y V_x)+\frac{\partial}{\partial z}(\rho V_z V_x) = -\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right]+\rho g_x$$

Assuming $\rho = \text{const}$, and multiplying both sides by $\frac{1}{\rho}$

$$\frac{\partial}{\partial t}(V_x)+\frac{\partial}{\partial x}(V_x V_x)+\frac{\partial}{\partial y}(V_y V_x)+\frac{\partial}{\partial z}(V_z V_x) = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}+\frac{\partial^2 V_x}{\partial z^2}\right]+g_x$$

Introducing Reynolds decomposition $V_i=\overline{V_i}+V_i'$ and $p=\overline{p}+p'$

Then, the resulting equation becomes:

$$\frac{\partial}{\partial t}(\overline{V_x}+V_x')+\frac{\partial}{\partial x}\left\{(\overline{V_x}+V_x')(\overline{V_x}+V_x')\right\}+\frac{\partial}{\partial y}\left((\overline{V_y}+V_y')(\overline{V_x}+V_x')\right)+\frac{\partial}{\partial z}\left((\overline{V_z}+V_z')(\overline{V_x}+V_x')\right)$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial x}(\overline{p}+p')+\frac{\mu}{\rho}\left[\frac{\partial^2}{\partial x^2}(\overline{V_x}+V_x')+\frac{\partial^2}{\partial y^2}(\overline{V_x}+V_x')+\frac{\partial^2}{\partial z^2}(\overline{V_x}+V_x')\right]+g_x$$

b) Time average the equation from part a) by using the averaging rules given below:

$$\begin{array}{llll} 1 & \overline{\frac{\partial \tilde{v}_i}{\partial j}} = \frac{\partial \overline{v_i}}{\partial j} & 4 & \overline{\overline{v_i}} = \overline{v_i} \\ 2 & \overline{\overline{v_i}v_i'} = 0 & 5 & \overline{v_i'} = 0 \\ 3 & \overline{\frac{\partial \tilde{p}}{\partial x}} = \frac{\partial \overline{p}}{\partial x} & 6 & \overline{p'} = 0 \end{array} \qquad (3)$$

where i can be x, y or z and j can be x, y, z or t.

$$\overline{V_x'V_y'}$$

$$\mu\left(\frac{\partial}{\partial}+\frac{\partial}{\partial}\right)+\mu^T\left(\frac{\partial}{\partial}+\frac{\partial}{\partial}\right)$$

Compare the resulting averaged equation expressed in terms of the mean variables with the basis un-averaged equation expressed in terms of the instantaneous variables and identify the new terms.

Time averaging term by term:

$$\overline{\frac{\partial}{\partial t}(\overline{V_x}+V_x')} = \overline{\frac{\partial}{\partial t}(\overline{V_x})}+\overline{\frac{\partial}{\partial t}(V_x')} \overset{1}{=} \frac{\partial}{\partial t}\Big(\overset{4\ \ \overline{V_x}}{\overline{\overline{V_x}}}\Big)+\frac{\partial}{\partial t}\Big(\overset{0\ \ 5}{\overline{V_x'}}\Big) = \frac{\partial}{\partial t}(\overline{V_x})$$

$$\overline{\frac{\partial}{\partial x}\left\{(\overline{V_x}+V_x')(\overline{V_x}+V_x')\right\}} \overset{1}{=} \frac{\partial}{\partial x}\left(\overline{\overline{V_x}\,\overline{V_x}+\overline{V_x}V_x'+V_x'\overline{V_x}+V_x'V_x'}\right) = \frac{\partial}{\partial x}\left(\overline{\overline{V_x}\,\overline{V_x}}+\overset{2\ \ 0}{\overline{\overline{V_x}V_x'}}+\overset{2\ \ 0}{\overline{V_x'\overline{V_x}}}+\overline{V_x'V_x'}\right) \overset{4}{=} \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{V_x}+\overline{V_x'V_x'}\right)$$

Similarily

$$\overline{\frac{\partial}{\partial y}\left((\overline{V_y}+V_y')(\overline{V_x}+V_x')\right)} = \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{V_x}+\overline{V_y'V_x'}\right)$$

$$\overline{\frac{\partial}{\partial z}\left((\bar{V}_z + V_z')(\bar{V}_x + V_x')\right)} = \frac{\partial}{\partial z}\left(\bar{V}_z \bar{V}_x + \overline{V_z' V_x'}\right)$$

$$\overline{-\frac{1}{\rho}\frac{\partial}{\partial x}(\bar{p}+p')} \overset{\rho = \text{const}}{=} -\frac{1}{\rho} \cdot \overline{\frac{\partial}{\partial x}(\bar{p}+p')} \overset{1}{=} -\frac{1}{\rho} \cdot \frac{\partial}{\partial x}\left(\overline{\bar{p}+p'}\right) = -\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\overline{\bar{p}}\right) + \frac{\partial}{\partial x}\left(\overset{\overset{6}{0}}{\overset{\shortparallel}{\overline{p'}}}\right)\right] \overset{3}{=} -\frac{1}{\rho}\frac{\partial}{\partial x}(\bar{p})$$

For all terms in the square brackets:

$$\overline{\frac{\mu}{\rho}\left(\frac{\partial^2}{\partial x^2}(\bar{V}_x + V_x')\right)} \underset{\substack{\uparrow \\ \rho \text{ and } \mu \\ \text{are constant}}}{=} \frac{\mu}{\rho} \cdot \left(\overline{\frac{\partial^2}{\partial x^2}(\bar{V}_x + V_x')}\right) \underset{\substack{\uparrow \\ \text{Applying 1} \\ \text{twice}}}{=} \frac{\mu}{\rho} \cdot \frac{\partial^2}{\partial x^2}\left(\overline{\bar{V}_x + V_x'}\right) = \frac{\mu}{\rho} \cdot \left(\frac{\partial^2}{\partial x^2}\left(\overset{4\ \bar{V}_x}{\overset{\shortparallel}{\overline{\bar{V}_x}}}\right) + \frac{\partial^2}{\partial x^2}\left(\overset{5\ 0}{\overset{\shortparallel}{\overline{V_x'}}}\right)\right) = \frac{\mu}{\rho} \cdot \frac{\partial^2}{\partial x^2}(\bar{V}_x)$$

This will be the case for the rest of the terms in the brackets as well.

The final term:

$\bar{g}_x = g_x$ as g_x is a constant.

The resuting equation is:

$$\frac{\partial \bar{V}_x}{\partial t} + \frac{\partial}{\partial x}\left(\bar{V}_x\bar{V}_x + \overline{V_x' V_x'}\right) + \frac{\partial}{\partial y}\left(\bar{V}_y\bar{V}_x + \overline{V_y' V_x'}\right) + \frac{\partial}{\partial z}\left(\bar{V}_z\bar{V}_x + \overline{V_z' V_x'}\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 \bar{V}_x}{\partial x^2} + \frac{\partial^2 \bar{V}_x}{\partial y^2} + \frac{\partial^2 \bar{V}_x}{\partial z^2}\right] + g_x$$

Comparing with the un-averaged equation in instantaneous variables reveals some new terms:

$$\frac{\partial}{\partial x}\left(\overline{V_x' V_x'}\right), \quad \frac{\partial}{\partial y}\left(\overline{V_y' V_x'}\right) \text{ and } \frac{\partial}{\partial z}\left(\overline{V_z' V_x'}\right)$$

These are the covariances of the fluctuations. Also known as "Reynold Stresses", they can be viewed as the turbulent viscosity (as an analog)

c) The turbulent momentum flux tensor, $\overline{\sigma}^{(t)}$, can be defined with components (only the components necessary to complete the exercise are given):

$$\overline{\sigma}^{(t)}_{xx} = \rho\overline{v'_x v'_x} \tag{4}$$

$$\overline{\sigma}^{(t)}_{yx} = \rho\overline{v'_y v'_x} \tag{5}$$

$$\overline{\sigma}^{(t)}_{zx} = \rho\overline{v'_z v'_x} \tag{6}$$

The turbulent momentum flux are not known, but empirical approximations might be used for $\overline{\sigma}^{(t)}$. By analogy to Newton's law of viscosity (i.e., employing the gradient- and Boussinesq hypotheses) the momentum fluxes might be written as (Jakobsen, 2014, page 122):

$$\overline{\sigma}^{(t)}_{xx} = -\mu^{(t)}\frac{\partial \overline{v}_x}{\partial x} \tag{7}$$

$$\overline{\sigma}^{(t)}_{yx} = -\mu^{(t)}\frac{\partial \overline{v}_x}{\partial y} \tag{8}$$

$$\overline{\sigma}^{(t)}_{zx} = -\mu^{(t)}\frac{\partial \overline{v}_x}{\partial z} \tag{9}$$

where $\mu^{(t)}$ is the turbulent viscosity.

Use the closures (7)-(9) to rewrite the equation derived in b) into a form that can be solved numerically.[1]

[1] For flow near walls, for example, the turbulent viscosity can be expressed as (e.g., Bird et al (2002), Transport phenomena, 2ed, Wiley, Eq (5.4-2), p. 163):

$$\mu^{(t)} = \mu\left(\frac{y\ v_*}{14.5\ \nu}\right)^3, \quad 0 < \frac{yv^*}{\nu} < 5 \tag{10}$$

where ν is the kinematic viscosity and v^* is the friction velocity. With an empirical relation for $\mu^{(t)}$, the Navier-Stokes equation for turbulent flows is closed as the equation can be expressed in terms of time average variables only. These unknowns can be determined by solving the transport equation as the number of unknowns equals the number of equations.

Combining equations (4)-(6) and (7)-(9) yields:

$$\rho\,\overline{V_x'V_x'} = \overline{\sigma}^{(t)}_{xx} = -\mu^{(t)}\frac{\partial \overline{V_x}}{\partial x} \Rightarrow \overline{V_x'V_x'} = -\frac{\mu^{(t)}}{\rho}\frac{\partial \overline{V_x}}{\partial x} \quad (1)$$

Similarily:

$$\overline{V_y'V_x'} = -\frac{\mu^{(t)}}{\rho}\frac{\partial \overline{V_x}}{\partial y} \quad (2)$$

$$\overline{V_z'V_x'} = -\frac{\mu^{(t)}}{\rho}\cdot\frac{\partial \overline{V_x}}{\partial z} \quad (3)$$

Looking at LHS of the equation from b):

$$\frac{\partial \overline{V_x}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{V_x} + \overline{V_x'V_x'}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{V_x} + \overline{V_y'V_x'}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{V_x} + \overline{V_z'V_x'}\right)$$

$$= \frac{\partial \overline{V_x}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{V_x}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{V_x}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{V_x}\right) + \underbrace{\frac{\partial}{\partial x}\left(\overline{V_x'V_x'}\right) + \frac{\partial}{\partial y}\left(\overline{V_y'V_x'}\right) + \frac{\partial}{\partial z}\left(\overline{V_z'V_x'}\right)}_{\text{Reynold stresses}}$$

Inserting (1), (2) and (3) into the Reynold stresses gives:

$$\frac{\partial}{\partial x}\left(\overline{V_x'V_x'}\right) + \frac{\partial}{\partial y}\left(\overline{V_y'V_x'}\right) + \frac{\partial}{\partial z}\left(\overline{V_z'V_x'}\right) = \frac{\partial}{\partial x}\left(-\frac{\mu^{(t)}}{\rho}\frac{\partial \overline{V_x}}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\mu^{(t)}}{\rho}\frac{\partial \overline{V_x}}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\mu^{(t)}}{\rho}\cdot\frac{\partial \overline{V_x}}{\partial z}\right)$$

Assuming ρ and $\mu^{(t)}$ is constant:

$$= -\frac{\mu^{(t)}}{\rho}\left[\frac{\partial^2}{\partial x^2}\overline{V_x} + \frac{\partial^2}{\partial y^2}\overline{V_x} + \frac{\partial^2}{\partial z^2}\overline{V_x}\right]$$

Inserting into the equation from b):

$$\frac{\partial \bar{V}_x}{\partial t} + \frac{\partial}{\partial x}\left(\bar{V}_x \bar{V}_x\right) + \frac{\partial}{\partial y}\left(\bar{V}_y \bar{V}_x\right) + \frac{\partial}{\partial z}\left(\bar{V}_z \bar{V}_x\right) - \frac{\mu^{(t)}}{\rho}\left[\frac{\partial^2}{\partial x^2}\bar{V}_x + \frac{\partial^2}{\partial y^2}\bar{V}_x + \frac{\partial^2}{\partial z^2}\bar{V}_x\right]$$

$$= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu}{\rho}\left[\frac{\partial^2 \bar{V}_x}{\partial x^2} + \frac{\partial^2 \bar{V}_x}{\partial y^2} + \frac{\partial^2 \bar{V}_x}{\partial z^2}\right] + g_x$$

Now, noticing that the expressions inside the square brackets are equal, we can collect the terms on RHS:

$$\boxed{\frac{\partial \bar{V}_x}{\partial t} + \frac{\partial}{\partial x}\left(\bar{V}_x \bar{V}_x\right) + \frac{\partial}{\partial y}\left(\bar{V}_y \bar{V}_x\right) + \frac{\partial}{\partial z}\left(\bar{V}_z \bar{V}_x\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\mu+\mu^{(t)}}{\rho}\left[\frac{\partial^2}{\partial x^2}\bar{V}_x + \frac{\partial^2}{\partial y^2}\bar{V}_x + \frac{\partial^2}{\partial z^2}\bar{V}_x\right] + g_x}$$

d) For non-reactive mixtures having constant fluid properties, the heat equation in terms of temperature can be written as:

$$\rho C_P\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] \qquad (11)$$

where C_P is specific heat and k is thermal conductivity.

The corresponding component mass balance equation for component A can be written as:

$$\left(\frac{\partial \omega_A}{\partial t} + \frac{\partial}{\partial x}(v_x\omega_A) + \frac{\partial}{\partial y}(v_y\omega_A) + \frac{\partial}{\partial z}(v_z\omega_A)\right) = D\left[\frac{\partial^2 \omega_A}{\partial x^2} + \frac{\partial^2 \omega_A}{\partial y^2} + \frac{\partial^2 \omega_A}{\partial z^2}\right] \qquad (12)$$

where ω_A is the mass fraction of species A and D is the binary diffusivity.

The temperature and species A mass fraction can be decomposed in an similar manner as the velocity and pressure. Insert the decomposed temperature, velocity and species mass fraction in the equations given above. Recall that the temperature equation has to be written on the flux form by use of the continuity equation before the turbulence modeling procedure can be applied.

Starting with the heat equation, we want to rewrite to the flux form using the continuity equation.

Looking at the flux form: $\frac{\partial}{\partial x}(V_x T) + \frac{\partial}{\partial y}(V_y T) + \frac{\partial}{\partial z}(V_z T)$

By use of the product rule:

$$= V_x\frac{\partial}{\partial x}(T) + T\frac{\partial}{\partial x}(V_x) + V_y\frac{\partial}{\partial y}(T) + T\frac{\partial}{\partial y}(V_y) + V_z\frac{\partial}{\partial z}(T) + T\frac{\partial}{\partial z}(V_z)$$

$$= V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} + V_z\frac{\partial T}{\partial z} + T\underbrace{\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right]}_{=\nabla\cdot V=0 \text{ by continuity}}$$

This means that

$$\frac{\partial}{\partial x}(V_x T)+\frac{\partial}{\partial y}(V_y T)+\frac{\partial}{\partial z}(V_z T) = V_x\frac{\partial T}{\partial x}+V_y\frac{\partial T}{\partial y}+V_z\frac{\partial T}{\partial z}$$

(or reverse)

Inserting into the heat equation, we get:

$$\rho C_p\left(\frac{\partial T}{\partial t}+\frac{\partial}{\partial x}(V_x T)+\frac{\partial}{\partial y}(V_y T)+\frac{\partial}{\partial z}(V_z T)\right) = k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]$$

Applying the decomposition of turbulent properties: $\psi = \overline{\psi}+\psi'$

$$\Rightarrow \rho C_p\left(\frac{\partial(\overline{T}+T')}{\partial t}+\frac{\partial\left((\overline{V_x}+V_x')(\overline{T}+T')\right)}{\partial x}+\frac{\partial\left((\overline{V_y}+V_y')(\overline{T}+T')\right)}{\partial y}+\frac{\partial\left((\overline{V_z}+V_z')(\overline{T}+T')\right)}{\partial z}\right)$$

$$= k\left[\frac{\partial^2(\overline{T}+T')}{\partial x^2}+\frac{\partial^2(\overline{T}+T')}{\partial y^2}+\frac{\partial^2(\overline{T}+T')}{\partial z^2}\right]$$

For the component balance, we can decompose directly:

$$\Rightarrow \frac{\partial(\overline{w_A}+w_A')}{\partial t}+\frac{\partial\left((\overline{V_x}+V_x')(\overline{w_A}+w_A')\right)}{\partial x}+\frac{\partial\left((\overline{V_y}+V_y')(\overline{w_A}+w_A')\right)}{\partial y}+\frac{\partial\left((\overline{V_z}+V_z')(\overline{w_A}+w_A')\right)}{\partial z}$$

$$= D\left[\frac{\partial^2(\overline{w_A}+w_A')}{\partial x^2}+\frac{\partial^2(\overline{w_A}+w_A')}{\partial y^2}+\frac{\partial^2(\overline{w_A}+w_A')}{\partial z^2}\right]$$

e) Time average the equations derived in part d) by using the averaging rules given below:

1. $\overline{\frac{\partial}{\partial i}(v_i T)} = \frac{\partial}{\partial i}(\overline{v_i T})$ $\qquad \overline{\frac{\partial}{\partial i}(v_i \omega_A)} = \frac{\partial}{\partial i}(\overline{v_i \omega_A})$ 5.

2. $\overline{\frac{\partial}{\partial t}(T)} = \frac{\partial}{\partial t}(\overline{T})$ $\qquad \overline{\frac{\partial}{\partial t}(\omega_A)} = \frac{\partial}{\partial t}(\overline{\omega_A})$ 6.

3. $\overline{v_i'\overline{T}} = 0$ $\qquad \overline{T'} = 0$ 7.

4. $\overline{v_i'\overline{\omega_A}} = 0$ $\qquad \overline{\omega_A'} = 0$ 8. (13)

where i can be x, y or z.

Compare the resulting time average equations with the initial instantaneous equations and identify any new terms.

Time averaging the temperature equation term by term:

- Assuming constant ρ and C_p, the LHS can be split:

$$\overline{\rho C_p\left(\frac{\partial(\overline{T}+T')}{\partial t}+\frac{\partial((\overline{V_x}+V_x')(\overline{T}+T'))}{\partial x}+\frac{\partial((\overline{V_y}+V_y')(\overline{T}+T'))}{\partial y}+\frac{\partial((\overline{V_z}+V_z')(\overline{T}+T'))}{\partial z}\right)}$$

$$=\rho C_p\left(\overline{\frac{\partial(\overline{T}+T')}{\partial t}}+\overline{\frac{\partial((\overline{V_x}+V_x')(\overline{T}+T'))}{\partial x}}+\overline{\frac{\partial((\overline{V_y}+V_y')(\overline{T}+T'))}{\partial y}}+\overline{\frac{\partial((\overline{V_z}+V_z')(\overline{T}+T'))}{\partial z}}\right)$$

Then, term by term:

$$\overline{\frac{\partial(\overline{T}+T')}{\partial t}}\overset{1}{=}\frac{\partial}{\partial t}\left(\overline{\overline{T}+T'}\right)=\frac{\partial}{\partial t}\left(\overline{\overline{T}}+\cancelto{7}{\overline{T'}}\right)=\frac{\partial\overline{T}}{\partial t}$$

$$\overline{\frac{\partial((\overline{V_x}+V_x')(\overline{T}+T'))}{\partial x}}=\frac{\partial}{\partial x}\left(\overline{\overline{V_x}\,\overline{T}}+\cancelto{3}{\overline{V_x'\overline{T}}}+\cancelto{3}{\overline{\overline{T}V_x'}}+\overline{V_x'T'}\right)=\frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{T}+\overline{V_x'T'}\right)$$

Expanding and using 1

Similarily, we get:

$$\overline{\frac{\partial((\overline{V_y}+V_y')(\overline{T}+T'))}{\partial y}}=\frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{T}+\overline{V_y'T'}\right)$$

$$\overline{\frac{\partial((\overline{V_z}+V_z')(\overline{T}+T'))}{\partial z}}=\frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{T}+\overline{V_z'T'}\right)$$

For RHS, assuming k to be constant

$$\overline{k\left[\frac{\partial^2(\overline{T}+T')}{\partial x^2}+\frac{\partial^2(\overline{T}+T')}{\partial y^2}+\frac{\partial^2(\overline{T}+T')}{\partial z^2}\right]}=k\left[\overline{\frac{\partial^2(\overline{T}+T')}{\partial x^2}}+\overline{\frac{\partial^2(\overline{T}+T')}{\partial y^2}}+\overline{\frac{\partial^2(\overline{T}+T')}{\partial z^2}}\right]$$

The terms are equal except for derivation variable

So for the first term:

$$\frac{\overline{\partial^2(\overline{T}+T')}}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\overline{\overline{T}+T'}\right) = \frac{\partial^2}{\partial x^2}\left(\overline{T} + \cancel{\overline{T'}}^{\,7}\right) = \frac{\partial^2 \overline{T}}{\partial x^2}$$

Similarily, for the final terms;

$$\frac{\overline{\partial^2(\overline{T}+T')}}{\partial y^2} = \frac{\partial^2 \overline{T}}{\partial y^2}$$

$$\frac{\overline{\partial^2(\overline{T}+T')}}{\partial z^2} = \frac{\partial^2 \overline{T}}{\partial z^2}$$

Inserting the time averaged terms into the original equation, finally gives:

$$\rho C_p\left(\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{T} + \overline{V_x'T'}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{T} + \overline{V_y'T'}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{T} + \overline{V_z'T'}\right)\right) = k\left[\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2} + \frac{\partial^2 \overline{T}}{\partial z^2}\right]$$

$\rho C_p \frac{\partial}{\partial x}\left(\overline{V_x'T'}\right)$, $\rho C_p \frac{\partial}{\partial y}\left(\overline{V_y'T'}\right)$ and $\rho C_p\left(\overline{V_z'T'}\right)$ are new terms

I have no idea what they are, could you comment that to me? 😅

Now, going back to the momentum equation, the LHS can be time averaged directly:

$$\overline{\frac{\partial(\overline{w_A}+w_A')}{\partial t} + \frac{\partial\left((\overline{V_x}+V_x')(\overline{w_A}+w_A')\right)}{\partial x} + \frac{\partial\left((\overline{V_y}+V_y')(\overline{w_A}+w_A')\right)}{\partial y} + \frac{\partial\left((\overline{V_z}+V_z')(\overline{w_A}+w_A')\right)}{\partial z}}$$

$$= \frac{\overline{\partial(\overline{w_A}+w_A')}}{\partial t} + \frac{\overline{\partial\left((\overline{V_x}+V_x')(\overline{w_A}+w_A')\right)}}{\partial x} + \frac{\overline{\partial\left((\overline{V_y}+V_y')(\overline{w_A}+w_A')\right)}}{\partial y} + \frac{\overline{\partial\left((\overline{V_z}+V_z')(\overline{w_A}+w_A')\right)}}{\partial z}$$

Again, term by term:

$$\frac{\overline{\partial(\overline{w_A}+w_A')}}{\partial t} = \frac{\partial}{\partial t}\left(\overline{\overline{w_A}+w_A'}\right) = \frac{\partial}{\partial t}\left(\overline{\overline{w_A}} + \cancel{\overline{w_A'}}^{\,8}\right) = \frac{\partial \overline{w_A}}{\partial t}$$

Noticing that the remainding terms on LHS are equal except for having x, y, z in the term, it is only necessary to show one of them

$$\frac{\partial \overline{\left((\bar{V}_x + V_x')(\bar{w}_A + w_A') \right)}}{\partial x} = \frac{\partial}{\partial x}\left(\overline{\bar{V}_x \bar{w}_A} + \cancelto{4}{\overline{V_x' \bar{w}_A}} + \cancelto{4}{\overline{\bar{V}_x w_A'}} + \overline{V_x' w_A'} \right) = \frac{\partial}{\partial x}\left(\bar{V}_x \bar{w}_A + \overline{V_x' w_A'} \right)$$

Similarily:

$$\frac{\partial \overline{\left((\bar{V}_y + V_y')(\bar{w}_A + w_A') \right)}}{\partial y} = \frac{\partial}{\partial y}\left(\bar{V}_y \bar{w}_A + \overline{V_y' w_A'} \right)$$

$$\frac{\partial \overline{\left((\bar{V}_z + V_z')(\bar{w}_A + w_A') \right)}}{\partial z} = \frac{\partial}{\partial z}\left(\bar{V}_z \bar{w}_A + \overline{V_z' w_A'} \right)$$

On RHS, assuming D is constant:

$$\overline{D\left[\frac{\partial^2 (\bar{w}_A + w_A')}{\partial x^2} + \frac{\partial^2 (\bar{w}_A + w_A')}{\partial y^2} + \frac{\partial^2 (\bar{w}_A + w_A')}{\partial z^2} \right]} = D\left[\frac{\partial^2 \overline{(\bar{w}_A + w_A')}}{\partial x^2} + \frac{\partial^2 \overline{(\bar{w}_A + w_A')}}{\partial y^2} + \frac{\partial^2 \overline{(\bar{w}_A + w_A')}}{\partial z^2} \right]$$

Again, be have 3 terms that only differ in if the coordinate variable is x, y or z.

$$\frac{\partial^2 \overline{(\bar{w}_A + w_A')}}{\partial x^2} = \frac{\partial^2 (\overline{\bar{w}_A} + \overline{w_A'})}{\partial x^2} = \frac{\partial^2 (\overline{\bar{w}_A} + \cancelto{8}{\overline{w_A'}})}{\partial x^2} = \frac{\partial^2 \bar{w}_A}{\partial x^2}$$

Similarily

$$\frac{\partial^2 \overline{(\bar{w}_A + w_A')}}{\partial y^2} = \frac{\partial^2 \bar{w}_A}{\partial y^2}$$

$$\frac{\partial^2 \overline{(\bar{w}_A + w_A')}}{\partial z^2} = \frac{\partial^2 \bar{w}_A}{\partial z^2}$$

Inserting the time averaged terms into the equation from d), gives:

$$\frac{\partial \bar{w}_A}{\partial t} + \frac{\partial}{\partial x}\left(\bar{V}_x \bar{w}_A + \overline{V_x' w_A'} \right) + \frac{\partial}{\partial y}\left(\bar{V}_y \bar{w}_A + \overline{V_y' w_A'} \right) + \frac{\partial}{\partial z}\left(\bar{V}_z \bar{w}_A + \overline{V_z' w_A'} \right) = D\left[\frac{\partial^2 \bar{w}_A}{\partial x^2} + \frac{\partial^2 \bar{w}_A}{\partial y^2} + \frac{\partial^2 \bar{w}_A}{\partial z^2} \right]$$

$\frac{\partial}{\partial x}\left(\overline{V_x' w_A'}\right)$, $\frac{\partial}{\partial y}\left(\overline{V_y' w_A'}\right)$, $\frac{\partial}{\partial z}\left(\overline{V_z' w_A'}\right)$ are new terms

I have no idea what they are, could you comment that to me? 😅

f) The turbulent heat flux, $\overline{q}^{(t)}$, can be defined with components:

$$\overline{q}_x^{(t)} = \rho C_P \overline{v_x' T'} \tag{14}$$

$$\overline{q}_y^{(t)} = \rho C_P \overline{v_y' T'} \tag{15}$$

$$\overline{q}_z^{(t)} = \rho C_P \overline{v_z' T'} \tag{16}$$

The turbulent heat flux are not known, but by analogy to Fourier law of heat conduction we may write:

$$\overline{q}_x^{(t)} = -k^{(t)} \frac{\partial \overline{T}}{\partial x} \tag{17}$$

$$\overline{q}_y^{(t)} = -k^{(t)} \frac{\partial \overline{T}}{\partial y} \tag{18}$$

$$\overline{q}_z^{(t)} = -k^{(t)} \frac{\partial \overline{T}}{\partial z} \tag{19}$$

where $k^{(t)}$ is the turbulent conductivity.

The turbulent mass flux, $\overline{j}^{(t)}$, can be defined with components:

$$\overline{j}_x^{(t)} = \rho \overline{v_x' \omega_A'} \tag{20}$$

$$\overline{j}_y^{(t)} = \rho \overline{v_y' \omega_A'} \tag{21}$$

$$\overline{j}_z^{(t)} = \rho \overline{v_z' \omega_A'} \tag{22}$$

The turbulent mass flux are not known, but by analogy to Fick's law of diffusion we may write:

$$\overline{j}_x^{(t)} = -\rho D^{(t)} \frac{\partial \overline{\omega_A}}{\partial x} \tag{23}$$

$$\overline{j}_y^{(t)} = -\rho D^{(t)} \frac{\partial \overline{\omega_A}}{\partial y} \tag{24}$$

$$\overline{j}_z^{(t)} = -\rho D^{(t)} \frac{\partial \overline{\omega_A}}{\partial z} \tag{25}$$

where $D^{(t)}$ is the turbulent diffusivity.

Use the expressions given to rewrite the equations derived in part e).

Combining (14)-(16) with (17)-(19) gives:

$$\rho C_p \overline{V_x' T'} = \overline{q}_x^{(t)} = -k^{(t)} \frac{\partial \overline{T}}{\partial x} \Rightarrow \overline{V_x' T'} = -\frac{k^{(t)}}{\rho C_p} \frac{\partial \overline{T}}{\partial x}$$

$$\rho C_p \overline{V_y' T'} = \overline{q}_y^{(t)} = -k^{(t)} \frac{\partial \overline{T}}{\partial y} \Rightarrow \overline{V_y' T'} = -\frac{k^{(t)}}{\rho C_p} \frac{\partial \overline{T}}{\partial y}$$

$$\rho C_p \overline{V_z' T'} = \overline{q}_z^{(t)} = -k^{(t)} \frac{\partial \overline{T}}{\partial z} \Rightarrow \overline{V_z' T'} = -\frac{k^{(t)}}{\rho C_p} \frac{\partial \overline{T}}{\partial z}$$

Inserting this into the heat equation from e):

For LHS:

$$\rho C_p \left(\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{T} + \overline{V_x' T'}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{T} + \overline{V_y' T'}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{T} + \overline{V_z' T'}\right)\right)$$

$$= \rho C_p \left(\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{T}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{T}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{T}\right)\right) + \rho C_p \left(\frac{\partial}{\partial x}\left(\overline{V_x' T'}\right) + \frac{\partial}{\partial y}\left(\overline{V_y' T'}\right) + \frac{\partial}{\partial z}\left(\overline{V_z' T'}\right)\right)$$

$$= \rho C_p \left(\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{T}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{T}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{T}\right)\right) + \rho C_p \left(\frac{\partial}{\partial x}\left(-\frac{k^{(t)}}{\rho C_p} \frac{\partial \overline{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{k^{(t)}}{\rho C_p} \frac{\partial \overline{T}}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{k^{(t)}}{\rho C_p} \frac{\partial \overline{T}}{\partial z}\right)\right)$$

Assuming that $-\frac{k^{(t)}}{\rho C_p}$ is constant, it can be drawn outside of the parenthesis

$$= \rho C_p \left(\frac{\partial \overline{T}}{\partial t} + \frac{\partial}{\partial x}\left(\overline{V_x}\,\overline{T}\right) + \frac{\partial}{\partial y}\left(\overline{V_y}\,\overline{T}\right) + \frac{\partial}{\partial z}\left(\overline{V_z}\,\overline{T}\right)\right) - k^{(t)}\left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2} + \frac{\partial^2 \overline{T}}{\partial z^2}\right)$$

The full equation then becomes:

These are equal

$$\rho C_p\left(\frac{\partial \bar{T}}{\partial t}+\frac{\partial}{\partial x}\left(\bar{V}_x\bar{T}\right)+\frac{\partial}{\partial y}\left(\bar{V}_y\bar{T}\right)+\frac{\partial}{\partial z}\left(\bar{V}_z\bar{T}\right)\right)-k^{(t)}\left(\frac{\partial^2\bar{T}}{\partial x^2}+\frac{\partial^2\bar{T}}{\partial y^2}+\frac{\partial^2\bar{T}}{\partial z^2}\right)=k\left[\frac{\partial^2\bar{T}}{\partial x^2}+\frac{\partial^2\bar{T}}{\partial y^2}+\frac{\partial^2\bar{T}}{\partial z^2}\right]$$

$$\rho C_p\left(\frac{\partial \bar{T}}{\partial t}+\frac{\partial}{\partial x}\left(\bar{V}_x\bar{T}\right)+\frac{\partial}{\partial y}\left(\bar{V}_y\bar{T}\right)+\frac{\partial}{\partial z}\left(\bar{V}_z\bar{T}\right)\right)=\left(k+k^{(t)}\right)\left[\frac{\partial^2\bar{T}}{\partial x^2}+\frac{\partial^2\bar{T}}{\partial y^2}+\frac{\partial^2\bar{T}}{\partial z^2}\right]$$

Combing (20) - (22) with (23) - (25):

$$\rho\,\overline{V_x' w_A'}=\bar{j}_x^{(t)}=-\rho D^{(t)}\frac{\partial \bar{w}_A}{\partial x}\Rightarrow \overline{V_x' w_A'}=-D^{(t)}\frac{\partial \bar{w}_A}{\partial x}$$

$$\rho\,\overline{V_y' w_A'}=\bar{j}_y^{(t)}=-\rho D^{(t)}\frac{\partial \bar{w}_A}{\partial y}\Rightarrow \overline{V_y' w_A'}=-D^{(t)}\frac{\partial \bar{w}_A}{\partial y}$$

$$\rho\,\overline{V_z' w_A'}=\bar{j}_z^{(t)}=-\rho D^{(t)}\frac{\partial \bar{w}_A}{\partial z}\Rightarrow \overline{V_z' w_A'}=-D^{(t)}\frac{\partial \bar{w}_A}{\partial z}$$

Inserting into the component balance from e):

$$\frac{\partial \bar{w}_A}{\partial t}+\frac{\partial}{\partial x}\left(\bar{V}_x\bar{w}_A-D^{(t)}\frac{\partial \bar{w}_A}{\partial x}\right)+\frac{\partial}{\partial y}\left(\bar{V}_y\bar{w}_A-D^{(t)}\frac{\partial \bar{w}_A}{\partial y}\right)+\frac{\partial}{\partial z}\left(\bar{V}_z\bar{w}_A-D^{(t)}\frac{\partial \bar{w}_A}{\partial z}\right)=D\left[\frac{\partial^2\bar{w}_A}{\partial x^2}+\frac{\partial^2\bar{w}_A}{\partial y^2}+\frac{\partial^2\bar{w}_A}{\partial z^2}\right]$$

Assumin $D^{(t)}$ to be independent of coordinates

$$\frac{\partial \bar{w}_A}{\partial t}+\frac{\partial}{\partial x}\left(\bar{V}_x\bar{w}_A\right)+\frac{\partial}{\partial y}\left(\bar{V}_y\bar{w}_A\right)+\frac{\partial}{\partial z}\left(\bar{V}_z\bar{w}_A\right)-D^{(t)}\left[\frac{\partial^2\bar{w}_A}{\partial x^2}+\frac{\partial^2\bar{w}_A}{\partial y^2}+\frac{\partial^2\bar{w}_A}{\partial z^2}\right]=D\left[\frac{\partial^2\bar{w}_A}{\partial x^2}+\frac{\partial^2\bar{w}_A}{\partial y^2}+\frac{\partial^2\bar{w}_A}{\partial z^2}\right]$$

Finally:

$$\frac{\partial \bar{w}_A}{\partial t}+\frac{\partial}{\partial x}\left(\bar{V}_x\bar{w}_A\right)+\frac{\partial}{\partial y}\left(\bar{V}_y\bar{w}_A\right)+\frac{\partial}{\partial z}\left(\bar{V}_z\bar{w}_A\right)=\left(D+D^{(t)}\right)\left[\frac{\partial^2\bar{w}_A}{\partial x^2}+\frac{\partial^2\bar{w}_A}{\partial y^2}+\frac{\partial^2\bar{w}_A}{\partial z^2}\right]$$

g) To be able to solve the equations derived in part f) numerically, the turbulent conductivity, $k^{(t)}$, and turbulent diffusivity, $D^{(t)}$, must be found. This is done by introducing two dimensionless numbers, the turbulent Prandtl and Schmidt number[2]:

$$Pr^{(t)} = \frac{\nu^t}{\alpha^t} = C_P \frac{\mu^{(t)}}{k^{(t)}} \tag{26}$$

$$Sc^{(t)} = \frac{\nu^t}{D^{(t)}} = \frac{\mu^t}{\rho D^{(t)}} \tag{27}$$

The turbulent viscosity is usually estimated from a turbulence model like equation (10). The turbulent diffusivities and conductivity can then be estimated as a function of the turbulent viscosity through the dimensionless numbers expressed in equations (26)-(27).

Replace the turbulent conductivity and the turbulent diffusivity in the equation from part f) with the expressions given above.

Rewriting (26): $k^{(t)} = C_p \frac{\mu^{(t)}}{Pr^{(t)}}$

(27): $D^{(t)} = \frac{\mu^{(t)}}{Sc^{(t)}}$

Inserting (26) into the heat equation from f:

$$\underline{\underline{\rho C_p \left(\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x}\left(\bar{V}_x \bar{T}\right) + \frac{\partial}{\partial y}\left(\bar{V}_y \bar{T}\right) + \frac{\partial}{\partial z}\left(\bar{V}_z \bar{T}\right)\right) = \left(k + C_p \frac{\mu^{(t)}}{Pr^{(t)}}\right)\left[\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\partial^2 \bar{T}}{\partial z^2}\right]}}$$

Inserting (27) into the component mass balance:

$$\underline{\underline{\frac{\partial \bar{w}_A}{\partial t} + \frac{\partial}{\partial x}\left(\bar{V}_x \bar{w}_A\right) + \frac{\partial}{\partial y}\left(\bar{V}_y \bar{w}_A\right) + \frac{\partial}{\partial z}\left(\bar{V}_z \bar{w}_A\right) = \left(D + \frac{\mu^{(t)}}{Sc^{(t)}}\right)\left[\frac{\partial^2 \bar{w}_A}{\partial x^2} + \frac{\partial^2 \bar{w}_A}{\partial y^2} + \frac{\partial^2 \bar{w}_A}{\partial z^2}\right]}}$$