Problem 1: Laminar flow in a tube

a) Start out from the handout note listing the governing equations in their rigorous form and simplify the continuity and momentum equations in cylindrical coordinates as much as possible for an incompressible Newtonian fluid having constant fluid properties (i.e., viscosity and density).

Assume ρ and μ = const

Cylindrical Coordinates(r, θ, z)

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad (3)$$

$$\Rightarrow \frac{\rho}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\rho}{r}\frac{\partial}{\partial \theta}(V_\theta) + \rho\frac{\partial V_z}{\partial z} = 0 \qquad /\cdot \frac{1}{\rho}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial v_z}{\partial z} = 0$$

$$\Rightarrow \nabla \cdot V = 0$$

Cylindrical Coordinates (r, θ, z)

r-component:

$$\frac{\partial}{\partial t}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta v_r) - \frac{\rho v_\theta^2}{r} + \frac{\partial}{\partial z}(\rho v_z v_r) = -\frac{\partial p}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta r}) + \frac{\sigma_{\theta\theta}}{r} - \frac{\partial}{\partial z}(\sigma_{zr}) + \rho g_r \qquad (9)$$

LHS ρ = const:

$$\rho\left[\frac{\partial V_r}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r V_r V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta V_r) - \frac{V_\theta^2}{r} + \frac{\partial}{\partial z}(V_z V_r)\right]$$

Chain rule

$$\rho\left[\frac{\partial V_r}{\partial t} + \underline{\frac{V_r}{r}\frac{\partial}{\partial r}(r V_r)} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \underline{\frac{V_r}{r}\frac{\partial V_\theta}{\partial \theta}} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z} + \underline{V_r\frac{\partial V_z}{\partial z}}\right]$$

Recognizing that the underlined terms can be grouped and cancelled due to mass continuity equation

$$\Rightarrow \frac{V_r}{r}\frac{\partial}{\partial r}(r V_r) + \frac{V_r}{r}\frac{\partial V_\theta}{\partial \theta} + V_r\frac{\partial V_z}{\partial z} = V_r\underbrace{\left[\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\cdot\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}\right]}_{=0} = 0$$

Finally, we get:

$$\rho\left[\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right]$$

RHS: $-\frac{\partial p}{\partial r} - \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr})} - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta r}) + \underbrace{\frac{\sigma_{\theta\theta}}{r}} - \frac{\partial}{\partial z}(\sigma_{zr}) + \rho g_r$

Following the hints:

Grouping $-\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\sigma_{\theta\theta}$

Chain rule: $-\frac{\partial \sigma_{rr}}{\partial r} - \frac{\sigma_{rr}}{r} + \frac{1}{r}\sigma_{\theta\theta}$

Inserting: $\sigma_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r} - \cancelto{0}{\frac{2}{3}(\nabla\cdot\mathbf{v})}\right]$ $\sigma_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \cancelto{0}{\frac{2}{3}(\nabla\cdot\mathbf{v})}\right]$

Previously, we found that $\nabla\cdot V = 0$

And applying $\mu = \text{const}$

$$2\mu\left[\frac{\partial}{\partial r}\underbrace{\left(\frac{\partial V_r}{\partial r} + \frac{V_r}{r}\right)} - \frac{1}{r}\cdot\frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r^2}\right]$$

Using the reverse chain rule, we recognize this to be

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r)$$

$$2\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right) - \frac{1}{r^2}\frac{\partial V_\theta}{\partial \theta}\right]$$

Back to the "full" RHS:

$$-\frac{\partial p}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta r}) - \frac{\partial}{\partial z}(\sigma_{zr}) + \rho g_r + 2\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right) - \frac{1}{r^2}\frac{\partial V_\theta}{\partial \theta}\right]$$

Inserting

$$\sigma_{r\theta} = \sigma_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$\mu = \text{const}$

$$-\frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta r}) = \frac{\mu}{r}\frac{\partial}{\partial \theta}\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$

$$= \frac{\mu}{r}\left[r\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial V_\theta}{\partial \theta}\right) + \frac{1}{r}\frac{\partial^2 V_r}{\partial \theta^2}\right]$$

$$= \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial V_\theta}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}\right]$$

And:

$$\sigma_{zr} = \sigma_{rz} = -\mu\left[\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right]$$

$$-\frac{\partial}{\partial z}(\sigma_{zr}) = \mu\left[\frac{\partial^2 V_z}{\partial r\,\partial z}+\frac{\partial^2 V_r}{\partial z^2}\right]$$

Back to full RHS:

$$-\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial V_\theta}{\partial\theta}\right)+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2}\right] + \mu\left[\frac{\partial^2 V_z}{\partial r\,\partial z}+\frac{\partial^2 V_r}{\partial z^2}\right] + \rho g_r + 2\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right) - \frac{1}{r^2}\frac{\partial V_\theta}{\partial\theta}\right]$$

We notice that these terms looks similar to $\nabla\cdot V$. Using that ∂ is distributive:

$$\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial V_\theta}{\partial\theta}\right)+\mu\frac{\partial}{\partial r}\left(\frac{\partial V_z}{\partial z}\right)+2\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)$$

$$=\mu\frac{\partial}{\partial r}\Big(\underbrace{\frac{1}{r}\cdot\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}+\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r)}_{=\nabla\cdot V=0}\Big)+\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right) = \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cdot V_r)\right)$$

We then end up with:

$$-\frac{\partial p}{\partial r}+\mu\cdot\frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2}+\mu\cdot\frac{\partial^2 V_r}{\partial z^2}+\rho g_r+\mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cdot V_r)\right)-2\mu\cdot\frac{1}{r^2}\frac{\partial V_\theta}{\partial\theta}$$

We can then separate out μ

$$-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r)\right)+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2}+\frac{\partial^2 V_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta}\right]+\rho g_r$$

Combining RHS and LHS:

r- component

$$\rho\left[\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\cdot\frac{\partial V_r}{\partial\theta}-\frac{V_\theta^2}{r}+V_z\frac{\partial V_r}{\partial z}\right] = -\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2}+\frac{\partial^2 V_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta}\right]+\rho g_r$$

θ-component:

$$\frac{\partial}{\partial t}(\rho v_\theta) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r v_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta v_\theta) + \frac{\rho v_r v_\theta}{r} + \frac{\partial}{\partial z}(\rho v_z v_\theta) = -\frac{1}{r}\frac{\partial p}{\partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{r\theta}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta\theta}) - \frac{\partial}{\partial z}(\sigma_{z\theta}) - \frac{(\sigma_{\theta r}-\sigma_{r\theta})}{r} + \rho g_\theta \tag{10}$$

LHS: ρ is constant

$$\rho\left[\frac{\partial v_\theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r v_\theta) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta v_\theta) + \frac{v_r v_\theta}{r} + \frac{\partial}{\partial z}(v_z v_\theta)\right]$$

Using the chain rule:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r v_\theta) = \frac{1}{r}\left[v_\theta \frac{\partial}{\partial r}(r v_r) + r v_r \frac{\partial v_\theta}{\partial r}\right] = \frac{v_\theta}{r}\frac{\partial}{\partial r}(r v_r) + v_r\frac{\partial v_\theta}{\partial r}$$

$$\frac{1}{r}\cdot\frac{\partial}{\partial \theta}(v_\theta v_\theta) = \frac{2}{r} v_\theta \frac{\partial v_\theta}{\partial \theta}$$

$$\frac{\partial}{\partial z}(v_z v_\theta) = v_z\frac{\partial v_\theta}{\partial z} + v_\theta\frac{\partial v_z}{\partial z}$$

Inserting into LHS:

$$\rho\left[\frac{\partial v_\theta}{\partial t} + \underline{\frac{v_\theta}{r}\frac{\partial}{\partial r}(r v_r)} + v_r\frac{\partial v_\theta}{\partial r} + \underline{\frac{2}{r} v_\theta\frac{\partial v_\theta}{\partial \theta}} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z} + \underline{v_\theta\frac{\partial v_z}{\partial z}}\right]$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial v_z}{\partial z} = 0$$

Recognizing that the underlined terms are equal to $v_\theta(\nabla\cdot v) = 0$

As we have a 2 in one of the terms we will not get rid of the term completely

Finally we get:

$$\underline{\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right]}$$

RHS

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{r\theta}) - \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} - \frac{\partial}{\partial z}(\sigma_{z\theta}) - \frac{\sigma_{\theta r}-\sigma_{r\theta}}{r} + \rho g_\theta$$

Applying chain rule:

$$-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sigma_{r\theta}) = -\frac{1}{r^2}\left(r^2\frac{\partial \sigma_{r\theta}}{\partial r} + \sigma_{r\theta}\frac{\partial r^2}{\partial r}\right) = -\frac{1}{r^2}\left(r^2\frac{\partial\sigma_{r\theta}}{\partial r} + \sigma_{r\theta}2r\right) = -\frac{\partial\sigma_{r\theta}}{\partial r} - \frac{2\sigma_{r\theta}}{r}$$

Inserting into RHS:

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} - \frac{\partial \sigma_{r\theta}}{\partial r} - \frac{2\sigma_{r\theta}}{r} - \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} - \frac{\partial}{\partial z}(\sigma_{z\theta}) - \frac{\sigma_{\theta r}-\sigma_{r\theta}}{r} + \rho g_\theta$$

$$\sigma_{r\theta} = \sigma_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

$$\Rightarrow \frac{\sigma_{\theta r} - \sigma_{r\theta}}{r} = 0 \qquad \sigma_{\theta r} = \sigma_{r\theta}$$

And:

$$-\frac{\partial \sigma_{r\theta}}{\partial r} - \frac{2\sigma_{r\theta}}{r} = -\frac{\partial}{\partial r}\left[-\mu\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\right] + \frac{2\mu}{r}\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$

μ is const

$$= \mu\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right) + \mu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right) + \frac{2\mu}{r}\left[r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$

Grouping terms together, and using that the partial derivative is distributive

$$=\mu\left[\frac{\partial}{\partial r}\left(\underline{r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)} + \underline{\frac{1}{r}\frac{\partial V_r}{\partial\theta}} + 2\frac{V_\theta}{r}\right) + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta}\right]$$

Using the chain rule on the underlined terms

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)\right) = \frac{\partial}{\partial r}\left(\cancel{r}\left(\frac{\frac{\partial V_\theta}{\partial r}\cdot r - V_\theta\cdot\cancel{\frac{\partial r}{\partial r}}^{1}}{r^{\cancel{2}}}\right)\right) = \frac{\partial}{\partial r}\left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{u(x)}{v(x)}\right) = \frac{u'\cdot v - u\cdot v'}{v^2}$$

continous function

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right) = -\frac{1}{r^2}\cdot\frac{\partial V_r}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial V_r}{\partial \theta}\right) = -\frac{1}{r^2}\cdot\frac{\partial V_r}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial V_r}{\partial r}\right)$$

We then insert the results back in:

$$-\frac{\partial\sigma_{r\theta}}{\partial r} - \frac{2\sigma_{r\theta}}{r} = \mu\left[\frac{\partial}{\partial r}\left(\frac{\partial V_\theta}{\partial r} - \cancel{\frac{V_\theta}{r}} + \cancel{2}\frac{V_\theta}{r}\right) \cancel{-\frac{1}{r^2}\cdot\frac{\partial V_r}{\partial\theta}} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial V_r}{\partial r}\right) + \frac{\cancel{2}^{1}}{r^2}\frac{\partial V_r}{\partial\theta}\right]$$

$$=\mu\left[\frac{\partial}{\partial r}\left(\frac{\partial V_\theta}{\partial r}+\frac{V_\theta}{r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial V_r}{\partial r}+\frac{V_r}{r}\right)\right]$$

Apply the inverse chain rule:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(rV_\theta) = \frac{\partial V_\theta}{\partial r}+\frac{V_\theta}{r}$$

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(rV_r) = \frac{\partial V_r}{\partial r}+\frac{V_r}{r}$$

$$=\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)\right]$$

Inserting into RHS

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)\right] - \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} - \frac{\partial}{\partial z}(\sigma_{z\theta}) + \rho g_\theta$$

$$\sigma_{\theta z} = \sigma_{z\theta} = -\mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]$$

Using that μ is constant, and that r is not a function of z

$$\Rightarrow -\frac{\partial}{\partial z}(\sigma_{z\theta}) = \mu\left[\frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r}\cdot\frac{\partial}{\partial z}\left(\frac{\partial V_z}{\partial \theta}\right)\right] = \mu\left[\frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial V_z}{\partial z}\right)\right]$$

Similarily:

0 due to $\nabla \cdot V = 0$

$$\sigma_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \cancel{\frac{2}{3}(\nabla\cdot\mathbf{v})}\right]$$

$$\Rightarrow -\frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} = \frac{2\mu}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{1}{r}\cdot\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right) = \frac{2\mu}{r}\left[\frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial V_\theta}{\partial \theta}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$

Inserting into RHS:

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)\right] + \frac{2\mu}{r}\left[\frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial V_\theta}{\partial \theta}\right) + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]$$
$$+ \mu\left[\frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\frac{\partial V_z}{\partial z}\right)\right] + \rho g_\theta$$

Using that μ is a common factor, and recognizing that multiple terms contains $\frac{1}{r}\cdot\frac{\partial}{\partial \theta}$, as well as using that $\frac{\partial}{\partial \theta}$ is distributive, we get:

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial \theta}\underbrace{\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(rV_r) + \frac{2}{r}\cdot\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}\right)}_{=\underbrace{\nabla\cdot V}_{=0} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}}\right] + \rho g_\theta$$

Finally RHS becomes:

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}\right] + \rho g_\theta$$

Combining LHS and RHS:

$$\boxed{\rho\left[\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{2}{r^2}\cdot\frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{1}{r^2}\cdot\frac{\partial^2 V_\theta}{\partial \theta^2}\right] + \rho g_\theta}$$

z-component:

$$\frac{\partial}{\partial t}(\rho v_z) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r v_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta v_z) + \frac{\partial}{\partial z}(\rho v_z v_z) = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta z}) - \frac{\partial}{\partial z}(\sigma_{zz}) + \rho g_z \tag{11}$$

LHS: Using $\rho = \text{const}$

$$\rho\left[\frac{\partial V_z}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rV_rV_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta V_z) + \frac{\partial}{\partial z}(V_zV_z)\right]$$

Expanding terms using the chain rule:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(rV_rV_z) = \frac{1}{r}\left[V_z\frac{\partial}{\partial r}(rV_r) + rV_r\frac{\partial V_z}{\partial r}\right] = \frac{V_z}{r}\cdot\frac{\partial}{\partial r}(rV_r) + V_r\frac{\partial V_z}{\partial r}$$

$$\frac{1}{r}\cdot\frac{\partial}{\partial \theta}(V_\theta V_z) = \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + \frac{V_z}{r}\cdot\frac{\partial V_\theta}{\partial \theta}$$

$$\frac{\partial}{\partial z}(V_zV_z) = 2V_z\frac{\partial V_z}{\partial z}$$

Inserting into LHS:

$$\rho\left[\frac{\partial V_z}{\partial t} + \underline{\frac{V_z}{r}\frac{\partial}{\partial r}(rV_r)} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\cdot\frac{\partial V_z}{\partial \theta} + \underline{\frac{V_z}{r}\frac{\partial V_\theta}{\partial \theta}} + \underline{2V_z\frac{\partial V_z}{\partial z}}\right]$$

The sum of the underlined terms is equal to $V_z \cdot \nabla\cdot V = 0$

The last term will "survive" due to the 2-factor

Finally LHS becomes:

$$\rho\left[\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right]$$

RHS

$$-\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta z}) - \frac{\partial}{\partial z}(\sigma_{zz}) + \rho g_z$$

Chain rule:

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) = -\frac{\partial \sigma_{rz}}{\partial r} - \frac{\sigma_{rz}}{r}$$

Using $\sigma_{zr} = \sigma_{rz} = -\mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$ and μ is const

Rewriting

$$= \mu\left[\frac{\partial}{\partial r}\left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right) + \frac{1}{r}\cdot\frac{\partial V_z}{\partial r} + \frac{1}{r}\frac{\partial V_r}{\partial z}\right]$$

swapping order of derivatives on some terms, and using the distributive property of derivation. AND using that r is not a function of z: $\frac{1}{r}\frac{\partial V_r}{\partial z} = \frac{\partial}{\partial z}\left(\frac{V_r}{r}\right)$

$$= \mu\left[\frac{\partial}{\partial z}\left(\frac{\partial V_r}{\partial r} + \frac{V_r}{r}\right) + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial V_z}{\partial r}\right)\right]$$

Using reverse chain rule:

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)\right) = \frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{r}\cdot\frac{\partial v_z}{\partial r} + \frac{\partial}{\partial r}\cdot\left(\frac{\partial v_z}{\partial r}\right)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}\right) = \mu\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r v_r\right)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

Inserting into RHS:

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v_z}{\partial r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\sigma_{\theta z}\right) - \frac{\partial}{\partial z}\left(\sigma_{zz}\right) + \rho g_z$$

Using $\sigma_{\theta z} = \sigma_{z\theta} = -\mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right]$, and $\mu = \text{const}$:

$$-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\sigma_{\theta z}\right) = \frac{\mu}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right)\right]$$

$$= \mu\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}\right]$$

?

Using $\sigma_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z} - \xcancel{\frac{2}{3}(\nabla\cdot\mathbf{v})}\right]$, and $\mu = \text{const}$

→ 0 from continuity eq

$$-\frac{\partial}{\partial z}\left(\sigma_{zz}\right) = 2\mu\frac{\partial^2 v_z}{\partial z^2}$$

Inserting into RHS and grouping on μ:

$$-\frac{\partial p}{\partial z} + \mu\left[\underline{\frac{\partial}{\partial z}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r v_r)\right)} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial v_z}{\partial r}\right) + \underline{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}\right)} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \underline{2\frac{\partial^2 v_z}{\partial z^2}}\right] + \rho g_z$$

The underlined terms are equal to $\frac{\partial}{\partial z}\left(\underbrace{\nabla\cdot V}_{=0}\right) = 0$

But the last term won't "die" due to the 2-factor

The result is:

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Combining LHS and RHS:

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta} + v_z\frac{\partial v_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

A liquid with density ρ and viscosity μ flows in a horizontal tube as shown in figure 1. The Reynolds number is so small that the flow can be considered laminar. The flow can be considered fully developed.

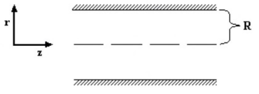


Figure 1: A horizontal tube.

b) Use the equation of motion (Navier-Stokes equation) and the equation of continuity in cylindrical coordinates (see handout of governing equations) to show that the radial velocity profile (the velocity in z direction as a function of r) can be written as:

$$v_z(r) = v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \qquad (1)$$

where v_{max} is the velocity in the middle of the tube (i.e., $v_{max} = v_z|_{r=0}$).

The following assumptions can be made in order to simplify the problem:

1. Steady state.
2. Fully developed flow thus neglect any entrance and outlet effects.
3. Axi-symmetric flow.
4. Convective transport only in z-direction (main direction of the flow).
5. Diffusive terms in the z-direction are negligible compared to the convective terms.
6. No gravity in z-direction because the tube is horizontal.
7. The pressure gradient in the flow direction, $\frac{dP}{dz}$, is constant

The meaning of the assumptions

1. $\frac{\partial}{\partial t} = 0$
2. $\frac{\partial v_z}{\partial z} = 0$
3. $\frac{\partial}{\partial \theta} = 0$
4. $v_r = v_\theta = 0$
5. 2nd derivatives with θ or $z \approx 0$
6. $\rho g_z = 0$
7. $\frac{\partial p}{\partial z} = \frac{dp}{dz} = \text{const} = k_1$

Moving forward, I will use $\nearrow^1$ to cross out terms that are zero due to assumption 1, $\nearrow^2$ for assumption 2 and so on. By beginning where I left off in a):

I realized that the exercise does not ask for r and θ, but I have already done them

r-component

$$\rho\left[\cancelto{1}{\frac{\partial v_r}{\partial t}} + \cancelto{4}{v_r\frac{\partial v_r}{\partial r}} + \cancelto{3}{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}} - \cancelto{4}{\frac{v_\theta^2}{r}} + \cancelto{4}{v_z\frac{\partial v_r}{\partial z}}\right] = -\frac{\partial p}{\partial r} + \mu\left[\cancelto{4}{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)} + \cancelto{3}{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \cancelto{4}{\frac{\partial^2 v_r}{\partial z^2}} - \cancelto{3}{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}\right] + \rho g_r$$

The only suriving term gives the result:

$$\frac{\partial p}{\partial r} = \rho g_r = 0$$

assuming negligible pressure drop in r-direction in order to get an equation system that can be solved analytically

θ-component

$$\rho\left[\cancelto{1}{\frac{\partial v_\theta}{\partial t}} + \cancelto{4}{v_r\frac{\partial v_\theta}{\partial r}} + \cancelto{4}{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{4}{\frac{v_r v_\theta}{r}} + \cancelto{4}{v_z\frac{\partial v_\theta}{\partial z}}\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\cancelto{4}{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)} + \cancelto{4}{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}} + \cancelto{4}{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancelto{4}{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}\right] + \rho g_\theta$$

$$\frac{\partial p}{\partial \theta} = r\rho g_\theta = 0$$

assuming negligible pressure drop in the θ-direction

Z-component

$$\rho\left[\overset{1}{\cancel{\frac{\partial v_z}{\partial t}}} + \overset{4}{\cancel{v_r \frac{\partial v_z}{\partial r}}} + \overset{4}{\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}} + \overset{2}{\cancel{v_z \frac{\partial v_z}{\partial z}}}\right] = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \overset{3}{\cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}} + \overset{2}{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}\right] + \overset{6}{\cancel{\rho g_z}}$$

We end up with

$$\frac{\partial p}{\partial z} = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

Using 7.: $\frac{\partial p}{\partial z} = \text{const} = k_1$

$$\Rightarrow \quad \frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{k_1}{\mu}\cdot r = k_2 \cdot r$$

Integration

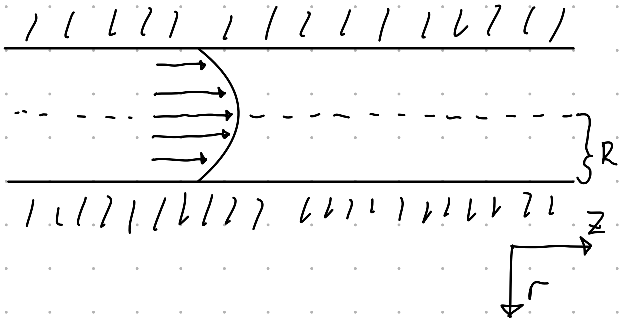
$$r\frac{\partial v_z}{\partial r} = \frac{1}{2}k_2 r^2 + k_3$$

$$\frac{\partial v_z}{\partial r} = \frac{1}{2}k_2 r + \frac{k_3}{r}$$

Integration

$$V_z = \frac{1}{4}k_2 r^2 + k_3 \ln r + k_4$$

Boundary conditions



$$V_z = V_{max}\Big|_{r=0} \Rightarrow k_4 + k_3\cdot\infty = V_{max}$$

$$\Rightarrow k_3 = 0,\ k_4 = V_{max}$$

$$\Rightarrow V_z = \frac{1}{4}k_2 r^2 + V_{max}$$

$$V_z = 0\Big|_{r=R} \Rightarrow \frac{1}{4}k_2 R^2 + V_{max} = 0$$

$$k_2 = -\frac{4V_{max}}{R^2}$$

$$\Rightarrow V_z = \frac{1}{\cancel{4}}\cdot\left(-\frac{\cancel{4}V_{max}}{R^2}\right)r^2 + V_{max}$$

$$= V_{max}\left(1 - \frac{r^2}{R^2}\right)$$

$$\boxed{V_z(z) = V_{max}\left(1 - \frac{r^2}{R^2}\right)}$$

c) Derive an expression for the cross-sectional average velocity using the velocity profile from part a. The constant maximum velocity, v_{max}, can be considered known.

The average velocity is found by:

$$\langle v_z \rangle = \frac{1}{A} \iint_A v_z(r)dA \tag{2}$$

For a pipe: $A = \pi R^2$

In cylinder coordinates, $dA = r\,dr\,d\theta$

$$\Rightarrow \langle V_z \rangle = \frac{1}{\pi R^2} \int_0^{2\pi}\int_0^R V_{max}\left(1 - \frac{r^2}{R^2}\right) r\,dr\,d\theta$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi} R^2} V_{max} \int_0^R r - \frac{r^3}{R^2}\,dr$$

$$= \frac{2V_{max}}{R^2}\left(\frac{1}{2}R^2 - \frac{1}{4}\frac{R^{\cancel{4}\,2}}{\cancel{R^2}}\right)$$

$$= \frac{\cancel{2}V_{max}}{\cancel{R^2}} \cdot \frac{1}{\cancel{4}\,2}\cancel{R^2}$$

$$\boxed{\langle V_z \rangle = \frac{V_{max}}{2}}$$

d) Plot the velocity and the shear stress profiles ~~by use of Matlab.~~

Using $R = 10$, $\mu = 3\ Pa\cdot s$, $V_{max} = 5\ m/s$

$$\Rightarrow d = \frac{2\mu}{R^2} V_{max} = \frac{2 \cdot 3}{10^2} \cdot 5 = 0{,}3\ N/m^3$$

$$\Rightarrow \sigma_{zr} = 0{,}3\ N/m^3 \cdot r$$

Need $\sigma_{zr}(r)$

$$\sigma_{zr} = -\mu\left(\frac{V_z}{\partial r} + \cancel{\frac{\partial V_r}{\partial z}}^{\,4}\right)$$

$$= -\mu \frac{\partial V_z}{\partial r}$$

$$= -\mu \frac{\partial}{\partial r}\left[V_{max}\left(1 - \frac{r^2}{R^2}\right)\right]$$

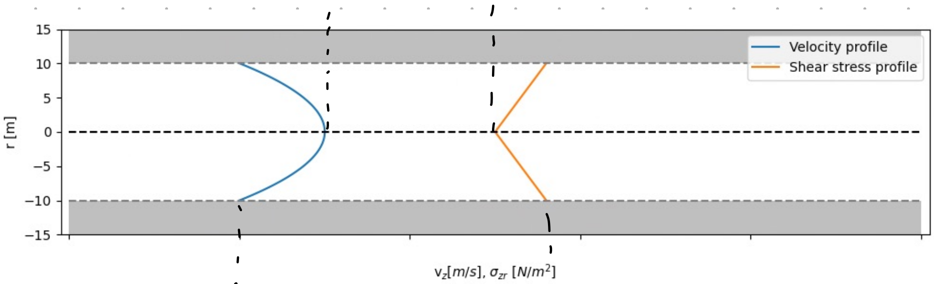
$$\underline{\sigma_{zr} = \frac{2\mu}{R^2} \cdot V_{max} \cdot r}$$

```python
import numpy as np
import matplotlib.pyplot as plt

L = 50 # Length
R = 10 # Radius of pipe
v_max = 5 # Speed of flow

# Arbitrary viscocity
my = 3

# Functions
def v_z(r):
    return v_max*(1 - (r/R)**2)

def s_zr(r):
    return 2*my/R**2 * v_max * r

#Performing calculations

r_values = np.linspace(-R, R, 100)
v_values = v_z(r_values) + 10 # +10 To set position of velocity profile
s_zr_values = s_zr(abs(r_values)) + 25 # +25 to set position of the profile
```

```python
fig, ax = plt.subplots(figsize=(10,3))
plt.plot(v_values, r_values, label='Velocity profile')
plt.plot(s_zr_values, r_values, label='Shear stress profile')
plt.plot([0,L], [R,R], color='grey', linestyle='--')
plt.plot([0,L], [-R,-R], color='grey', linestyle='--')
plt.plot([0,L], [0,0], color='black', linestyle='--')
plt.ylim(-R-5, R+5)
plt.xlim(0-0.5, L+0.5)
ax.fill_between([0,L], R, R+6, alpha=.5, linewidth=0, color='grey')
ax.fill_between([0,L], -R, -R-6, alpha=.5, linewidth=0, color='grey')
plt.xlabel('z')
plt.ylabel('r')
plt.legend()
plt.tight_layout()
plt.show()
```

The resulting profile is:

$r=0$, $V_z = V_{z,max}$ $\quad$ $r=0$, $\sigma_{zr} = 0$



$r = R$, $V_z = V_{z,max}$ $\quad$ $r = R$, $\sigma_{zr} = \sigma_{zr,max} = 3\ N/m^2$

e) In many biological applications, the biofluids' response to the shear deformation cannot be described by the Newton's law viscosity, i.e., their viscosities are not constant but functions of the shear rate. One simple yet useful model used for this type of fluids is the power law model:

$$\boldsymbol{\sigma} = -\eta(\dot{\gamma})\,\dot{\boldsymbol{\gamma}} \qquad \text{and} \qquad \mu \approx \eta(\dot{\gamma}) = m\dot{\gamma}^{n-1} \tag{3}$$

where m, n are the model parameters, $\boldsymbol{\sigma}$ is the deviatoric stress, $\dot{\boldsymbol{\gamma}} = \nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathsf{T}}$ and its magnitude, $\dot{\gamma}$, is given by

$$\dot{\gamma} = \sqrt{\frac{1}{2}\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}} \tag{4}$$

By considering the physical configuration and the assumptions in parts a) and b) (except constant viscosity), show that the rz component of σ is

$$\sigma_{rz} = m\left(-\frac{dv_z}{dr}\right)^n \tag{5}$$

$$\sigma = -\eta(\dot{\gamma})\dot{\gamma} = -(m\,\dot{\gamma}^{n-1})\dot{\gamma} = -(m\,\dot{\gamma}^{n-1})\left[\nabla V + (\nabla V)^T\right]$$

Using that any position in cylinder coordinates can be written as: (r, θ, z)

We know that:

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

And

$$\nabla V = \left(\frac{\partial}{\partial r}, \frac{1}{r}\cancelto{3}{\frac{\partial}{\partial \theta}}, \frac{\partial}{\partial z}\right)\left(\cancelto{4}{V_r}, \cancelto{4}{V_\theta}, V_z\right) = \left(\frac{\partial}{\partial r}, 0, \frac{\partial}{\partial z}\right)(0, 0, V_z)$$

Doing the vector calculations, we get more terms, however, due to the assumptions, they can be neglected.

$$\nabla V = \begin{bmatrix} 0 & 0 & \frac{\partial V_z}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & \cancelto{2}{\frac{\partial V_z}{\partial z}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{dV_z}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \nabla V^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{dV_z}{dr} & 0 & 0 \end{bmatrix}$$

Then: $$\dot{\gamma} = \nabla V + (\nabla V)^T = \begin{bmatrix} 0 & 0 & \frac{dV_z}{dr} \\ 0 & 0 & 0 \\ \frac{dV_z}{dr} & 0 & 0 \end{bmatrix}$$

$$\dot{\gamma}:\dot{\gamma} = \left(\frac{\partial V_z}{\partial r}\right)^2 + \left(\frac{\partial V_z}{\partial r}\right)^2 = 2\left(\frac{\partial V_z}{\partial r}\right)^2$$

$$\dot{\gamma} = \sqrt{\frac{1}{2}\dot{\gamma}:\dot{\gamma}} = \sqrt{\left(\frac{\partial V_z}{\partial r}\right)^2} = \left|\frac{\partial V_z}{\partial r}\right|$$

$$\sigma = -(m\dot{\gamma}^{n-1})\left[\nabla V + (\nabla V)^T\right] = -m\left|\frac{\partial V_z}{\partial r}\right|^{n-1} \begin{bmatrix} 0 & 0 & \frac{\partial V_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial V_z}{\partial r} & 0 & 0 \end{bmatrix}$$

As we know $\frac{\partial V_z}{\partial r}$ is negative (looking at the velocity profile) $\left|\frac{\partial V_z}{\partial r}\right|$ must be equal to $\left(-\frac{\partial V_z}{\partial r}\right)$

Then: $-m\left|\frac{\partial V_z}{\partial r}\right|^{n-1} \frac{\partial V_z}{\partial r} = m\left(-\frac{\partial V_z}{\partial r}\right)^{n-1}\left(-\frac{\partial V_z}{\partial r}\right) = m\left(-\frac{\partial V_z}{\partial r}\right)^n$

$$\sigma = \begin{bmatrix} 0 & 0 & m\left(-\frac{\partial V_z}{\partial r}\right)^n \\ 0 & 0 & 0 \\ m\left(-\frac{\partial V_z}{\partial r}\right)^n & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

Which then means that:

$$\boxed{\sigma_{rz} = m\left(-\frac{\partial V_z}{\partial r}\right)^n}$$

f) Repeat part b) for a power law fluid to show that the velocity profile is

$$v_z(r) = \left[\left(-\frac{dP}{dz}\right)\frac{R}{2m}\right]^{1/n} \frac{R}{(1/n)+1}\left[1-\left(\frac{r}{R}\right)^{(1/n)+1}\right] \qquad (6)$$

and recover the velocity profile for the Newtonian case given in part b), by setting $n = 1$ in Eq. (6).

We need to start from the same starting point as in a), however, the LHS is the same as we got in part a)

z-component:

$$\frac{\partial}{\partial t}(\rho v_z) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r v_z) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta v_z) + \frac{\partial}{\partial z}(\rho v_z v_z) =$$
$$-\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) - \frac{1}{r}\frac{\partial}{\partial \theta}(\sigma_{\theta z}) - \frac{\partial}{\partial z}(\sigma_{zz}) + \rho g_z$$

$$\text{LHS}: \rho\left[\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right]$$

for RHS, from e), we found that

$$\sigma = \begin{bmatrix} 0 & 0 & m\left(-\frac{\partial V_z}{\partial r}\right)^n \\ 0 & 0 & 0 \\ m\left(-\frac{\partial V_z}{\partial r}\right)^n & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$

Which means that $\sigma_{\theta z} = \sigma_{zz} = 0$

RHS then becomes:

$$-\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \rho g_z$$ Using the result from e)

Combining LHS and RHS

$$\rho\left[\cancel{\frac{\partial V_z}{\partial t}}^{1} + \cancel{V_r\frac{\partial V_z}{\partial r}}^{4} + \cancel{\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}}^{4} + V_z\cancel{\frac{\partial V_z}{\partial z}}^{2}\right] = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \cancel{\rho g_z}^{6}$$ Using the assumptions from b)

$$\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) = -\frac{\partial p}{\partial z}$$

$$\frac{\partial}{\partial r}(r\sigma_{rz}) = \left(-\frac{\partial p}{\partial z}\right) \quad \Big/ \text{integration (remembering that } \frac{\partial p}{\partial z} \text{ is constant)}$$

$$r\sigma_{rz} = \left(-\frac{\partial p}{\partial z}\right)\frac{r^2}{2} + C_1$$

$$\sigma_{rz} = \left(-\frac{\partial p}{\partial z}\right)\frac{r}{2} + \frac{C_1}{r}$$

As the flow is finite, and $r=0 \Rightarrow \infty$, C_1 must be 0

$$\sigma_{rz} = \left(-\frac{\partial p}{\partial z}\right)\frac{r}{2}$$

Using the result from f)

$$m\left(-\frac{\partial V_z}{\partial r}\right)^n = \left(-\frac{\partial p}{\partial z}\right)\frac{r}{2}$$

$$\frac{\partial V_z}{\partial r} = -\left(\left(-\frac{\partial p}{\partial z}\right)\frac{r}{2m}\right)^{1/n} = -\left(\left(-\frac{\partial p}{\partial z}\right)\frac{1}{2m}\right)^{1/n} \cdot r^{1/n}$$

Integrating

$$V_z = -\left(\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{1}{2m}\right)^{1/n} \cdot r^{(1+1/n)} \cdot \frac{1}{1+1/n} + C_2$$

Assuming $V_z = 0$ when $r=R$

$$\Rightarrow \quad V_z = -\left(\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{1}{2m}\right)^{1/n} R^{(1+1/n)} \cdot \frac{1}{1+1/n} + C_2 = 0$$

$$\Rightarrow C_2 = \left(\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{1}{2m}\right)^{1/n} R^{(1+1/n)} \cdot \frac{1}{1+1/n}$$

Then, we get:

$$V_z(r) = \left[\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{1}{2m}\right]^{1/n} \cdot \frac{1}{1+1/n}\cdot\left[R^{(1+1/n)} - r^{(1+1/n)}\right]$$

$$V_z(r) = \left[\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{1}{2m}\right]^{1/n} \cdot \frac{R^{(1+1/n)}}{1+1/n}\cdot\left[1 - \left(\frac{r}{R}\right)^{(1+1/n)}\right]$$

$$V_z(r) = \left[\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{R}{2m}\right]^{1/n} \cdot \frac{R}{1+1/n}\cdot\left[1 - \left(\frac{r}{R}\right)^{(1+1/n)}\right]$$

Inserting $n=1$:

$$V_z(r) = \left[\left(-\frac{\partial p}{\partial z}\right)\cdot\frac{R}{2m}\right]\cdot\frac{R}{1+1}\cdot\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$= \left(-\frac{\partial p}{\partial z}\right)\cdot\frac{R^2}{4m}\left[1-\left(\frac{r}{R}\right)^2\right]$$

Looking at b), and seeing what the constants are by going backwards, we can see that

Looking at b), and seeing what the constants are by going backwards, and as $n=1 \Rightarrow \mu = m \cdot \dot{\gamma}^{n-1} = m \cdot \dot{\gamma}^{0} = m$
$\Rightarrow \mu = m$

We can find that $V_{max} = -\frac{R^2}{4} k_2 = -\frac{R^2}{4\mu} \cdot \frac{\partial p}{\partial z} = \left(-\frac{\partial p}{\partial z}\right) \cdot \frac{R^2}{4\mu} = \left(-\frac{\partial p}{\partial z}\right) \cdot \frac{R^2}{4m}$

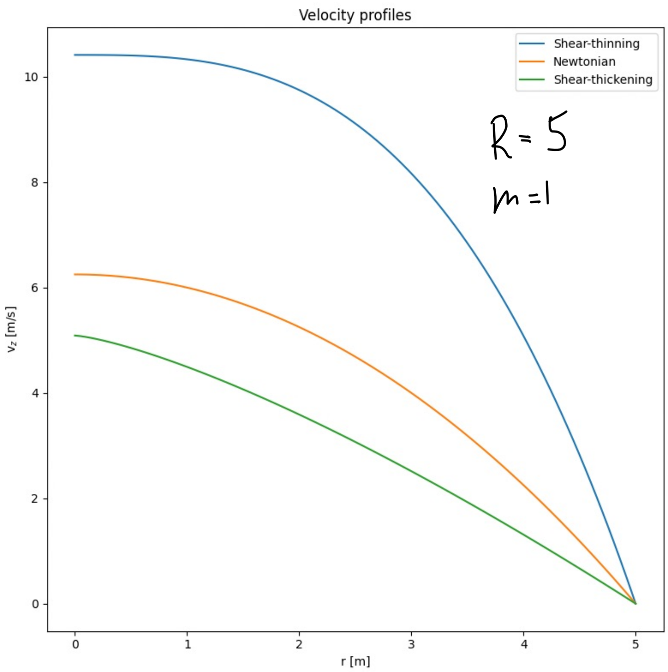
Finally:

$$V_z(r) = V_{max} \cdot \left[1 - \left(\frac{r}{R}\right)^2\right]$$, which is equal to what we found i b)

g) By taking $m = 1$, $\frac{dP}{dz} = -1$, $R = 5$, plot the velocity profiles for shear-thinning ($n = 0.5$), Newtonian ($n = 1$) and shear-thickening ($n = 3$) cases. At what positions the apparent viscosity has its largest and smallest values in each case? Comment based on the shape of the velocity profiles.

Reformulating using $\frac{dP}{dz} = k_1 = -1$, and plotting in python

$$V_z(r) = \left[\frac{-k_1 R}{2m}\right]^{1/n} \cdot \frac{R}{1 + 1/n} \cdot \left[1 - \left(\frac{r}{R}\right)^{(1+1/n)}\right]$$



The viscocity is given by

$$\mu(\dot{\gamma}) = m \cdot \dot{\gamma}^{n-1} = m \cdot \left|\frac{\partial V_z}{\partial r}\right|^{n-1} \underset{m=1}{=} \left|\frac{\partial V_z}{\partial r}\right|^{n-1}$$

So, for large $\left|\frac{\partial V_z}{\partial r}\right|$, the viscocity is largest for high values of n, and decreasing with decreasing n. This means that the steepest curve represents the largest viscocity.

So the viscocity is largest for shear-thinning, then newtonian and the smallest is for shear-thickening. For $n=1$, the viscocity is "independent" of $\left|\frac{\partial V_z}{\partial r}\right|$. It makes sense that shear-thinning has the highest velocity, as it has the lowest viscocity $\mu \propto \left|\frac{\partial V_z}{\partial r}\right|^{0,5}$, which in turn means that the shear stress of the wall is lower, and the effect of the shear stress decreases quicker (which is the reason for the steep slope)

This is also why the profile is flatter on the top with decreasing n. The decreasing μ, means that the effect of the shear stress from the wall decreases quicker while moving to $r=0$ than for larger n-values. This allows the fluid to retain V_{max} (or close to V_{max}) further away from $r=0$ than for large n.

From the shapes, it appears that the largest apparent viscocities are close to the pipe walls, as $\left|\frac{\partial v_z}{\partial r}\right|$ has the largest value there (steepest decline), and that the smallest viscocity is in the middle of the pipe, where $\left|\frac{\partial z}{\partial r}\right|$ has the smallest value