# 1 Model approximation (Half-rule)

For the following four processes

1. Find first and second order approximations using the half rule

   (a)
   $$G_1(s) = \frac{1}{(5s+1)(1.8s+1)} \tag{1}$$

   (b)
   $$G_2(s) = \frac{1}{(5s+1)(1.8s+1)(0.8s+1)} \tag{2}$$

   (c)
   $$G_3(s) = \frac{-0.5s+1}{(5s+1)(1.8s+1)(0.8s+1)} \tag{3}$$

   (d)
   $$G_4(s) = \frac{e^{-0.5s}}{(5s+1)(1.8s+1)(0.8s+1)^2} \tag{4}$$

   (e)
   $$G(s) = \frac{1}{s(3s+1)(s+1)} \tag{5}$$

   Hint: The integral term can be seen as a first-order model with a very long time constant, that is
   $$\frac{1}{s} \approx \frac{T}{Ts+1}, \quad T \to \infty \tag{6}$$

a) 1st order: $\theta = \theta_0 + \frac{\tau_2}{2} = \frac{\tau_2}{2} = \frac{1,8}{2} = 0,9$

$\tau_I = \tau_1 + \frac{\tau_2}{2} = 5 + 0,9 = 5,9$

$$\underline{\underline{G_{1,1}(s) = \frac{1}{(5,9s+1)} e^{-0,9s}}}$$

2nd order: (Already 2nd order)

$$\underline{\underline{G_{1,2}(s) = \frac{1}{(5s+1)(1,8s+1)}}}$$

b) 1st order: $\theta = \theta_0 + \frac{\tau_2}{2} + \tau_3 = \frac{1,8}{2} + 0,8 = 1,7$

$\tau_I = \tau_1 + \frac{\tau_2}{2} = 5 + 0,9 = 5,9$

$$\underline{\underline{G_{2,1}(s) = \frac{1}{5,9s+1} e^{-1,7s}}}$$

2nd order: Keep largest factor, $\theta = \theta_0 + \frac{\tau_3}{2} = \frac{0,8}{2} = 0,4$

$\tau_{II} = \tau_2 + \frac{\tau_3}{2} = 1,8 + 0,4 = 2,2$

$$\underline{\underline{G_{2,2}(s) = \frac{1}{(5s+1)(2,2s+1)} e^{-0,4s}}}$$

c) 1st order: $\theta = \theta_0 + \frac{\tau_2}{2} + \tau_3 + T_1 = 0 + 0{,}9 + 0{,}8 + 0{,}5 = 2{,}2$

$$(-0{,}5s+1) \approx e^{-0{,}5 \cdot s}$$

$$\tau_I = \tau_1 + \frac{\tau_2}{2} = 5 + 0{,}9 = 5{,}9$$

$$\underline{\underline{G_{3,1}(s) = \frac{1}{5{,}9s+1} e^{-2{,}2s}}}$$

2nd order: Keep $\frac{1}{5s+1}$, $\theta = \theta_0 + \frac{\tau_3}{2} = 0{,}5 + \frac{0{,}8}{2} = 0{,}9$

$$\tau_I = \tau_2 + \frac{\tau_3}{2} = 1{,}8 + \frac{0{,}8}{2} = 2{,}2$$

$$\underline{\underline{G_{3,2}(s) = \frac{1}{(5s+1)(2{,}2s+1)} \cdot e^{-0{,}9s}}}$$

d) 1st order: $\theta = \theta_0 + \frac{\tau_2}{2} + 2 \cdot \tau_3 = 0{,}5 + 0{,}9 + 2 \cdot 0{,}8 = 3$

$$\tau_I = \tau_1 + \frac{\tau_2}{2} = 5 + 0{,}9 = 5{,}9$$

$$\underline{\underline{G_{4,1}(s) = \frac{1}{5{,}9s+1} e^{-3s}}}$$

2nd order: $\theta = \theta_0 + \frac{\tau_3}{2} + \tau_3 = 0{,}5 + 0{,}4 + 0{,}8 = 1{,}7$

$$\tau_I = \tau_2 + \frac{\tau_3}{2} = 1{,}8 + 0{,}4 = 2{,}2$$

$$\underline{\underline{G_{4,2}(s) = \frac{1}{(5s+1)(2{,}2s+1)} e^{-1{,}7s}}}$$

e) If $\frac{1}{s}$ is approximated by $\frac{T}{Ts+1}$ $T \to \infty$, then T is the highest time constant

$\Rightarrow$ $\frac{1}{s}$ is "kept" instead of the other terms

1st order: $\theta = \theta_0 + \frac{\tau_1}{2} + \tau_2 = 0 + 1,5 + 1 = 2,5$

$\tau_I = T + \frac{\tau_1}{2}$, if $T \to \infty$ $\tau_I = T = \infty$

$$\underline{\underline{G_{5,1}(s) = \frac{1}{s} e^{-2,5s}}}$$

2nd order: $\theta = \theta_0 + \frac{\tau_2}{2} = \frac{1}{2} = 0,5$

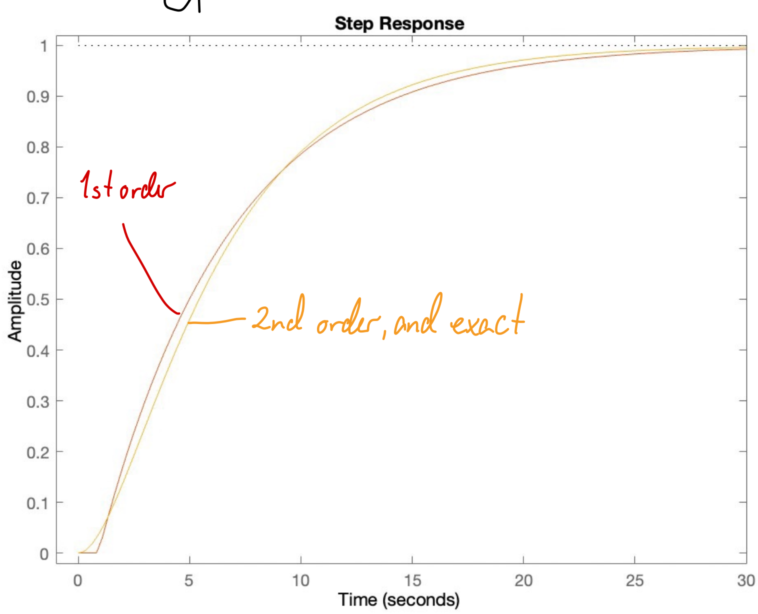
$\tau_I = \tau_1 + \frac{\tau_2}{2} = 3 + \frac{1}{2} = 3,5$

$$\underline{\underline{G_{5,2}(s) = \frac{1}{s(3,5s+1)} e^{-0,5s}}}$$

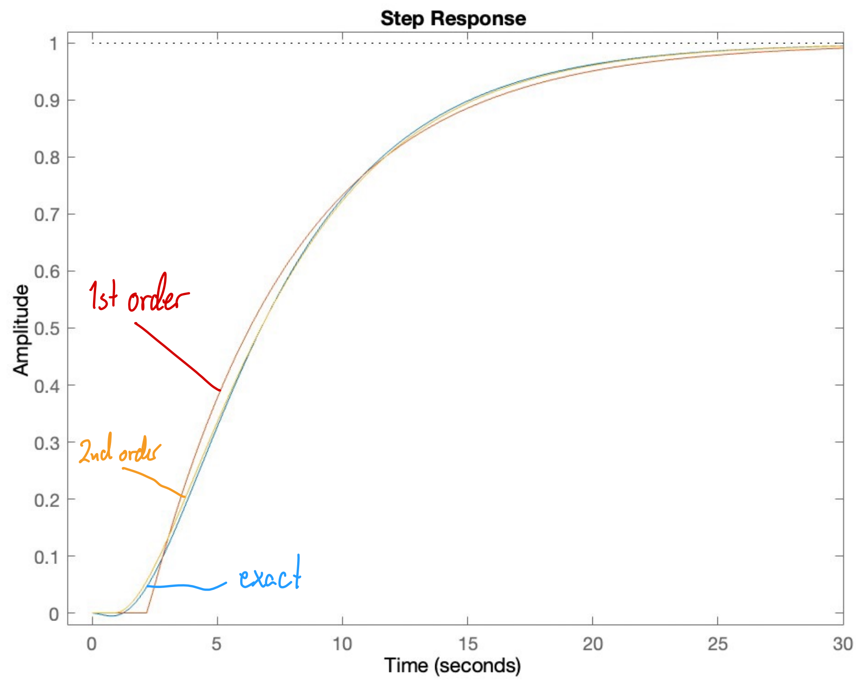
2. Plot the step response of your approximations of $G_1(s)$ and $G_3(s)$ into the plots in Figure 1. Use the `step` function in Matlab.

```
s = tf('s');

g1 = 1/((5*s + 1)*(1.8*s + 1));
g1_1 = exp(-0.9*s)/(5.9*s + 1);
g1_2 = 1/((5*s + 1)*(1.8*s + 1));

step(g1, g1_1, g1_2)
axis([-1 30 -0.02 1.02])

g3 = (-0.5*s + 1)/((5*s + 1)*(1.8*s + 1)*(0.8*s + 1));
g3_1 = exp(-2.2*s)/(5.9*s + 1);
g3_2 = exp(-0.9*s)/((5*s + 1)*(2.2*s + 1));

step(g3, g3_1, g3_2)
axis([-1 30 -0.02 1.02])
```

$G_1$



$G_3$




## 2 Closed loop transfer function

Consider the closed loop system given in Figure 2.

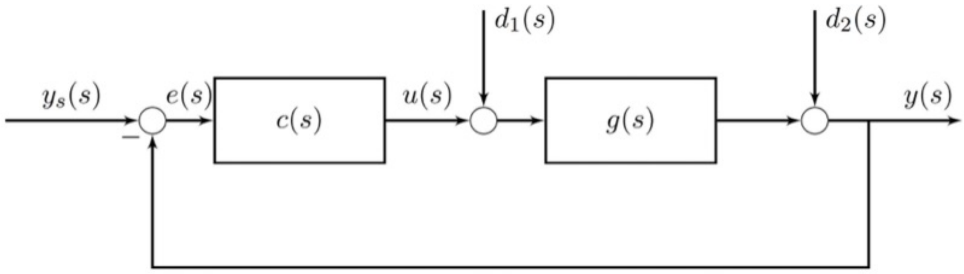



Figure 2: Controlled System

Let the transfer function of the plant be

$$g(s) = K\frac{1}{\tau_1 s + 1}. \tag{7}$$

The values for the plant parameters are $K = 4$ and $\tau_1 = 0.5$. The transfer function of the controller (PI) is

$$c(s) = \frac{K_c}{\tau_I s}\left(\tau_I s + 1\right). \tag{8}$$

The controller is tuned using the SIMC tuning rules with $\tau_c = 0.25$. Since $\theta = 0$, we have

$$K_c = \frac{1}{K}\frac{\tau_1}{\tau_c + \theta} = \frac{1}{4}\frac{0.5}{(0.25 + 0)} = 0.5 \tag{9}$$

and

$$\tau_I = \min\left\{\tau_1\, 4(\tau_c + \theta)\right\} = \min\left\{0.5,\, 4(0.25 + 0)\right\} = \tau_1 = 0.5. \tag{10}$$

### Tasks

Write the following transfer functions in the time-constant form:

1. $T(s)$: The transfer function from $y_s(s)$ to $y(s)$.
2. $M(s)$: The transfer function from $d_1(s)$ to $y(s)$.
3. $S(s)$: The transfer function from $d_2(s)$ to $y(s)$.

Transfer function for closed loop: $T_{CL} = \frac{T_{direct\ path}}{1 + T_{entire\ loop}}$

Inserting parameters into g(s) and c(s)

$$g(s) = \frac{4}{0{,}5s + 1}$$

$$c(s) = \frac{0{,}5}{0{,}5 \cdot s}(0{,}5s+1) = \frac{1}{s}(0{,}5s+1)$$

$$c \cdot g = \frac{4}{\cancel{0{,}5s+1}} \cdot \frac{1}{s} \cdot \cancel{(0{,}5s+1)} = \frac{4}{s}$$

1. $T_{dir} = c \cdot g$

$T_{loop} = c \cdot g$

$$\Rightarrow T(s) = \frac{c \cdot g}{1 + c \cdot g} = \frac{4/s}{1 + 4/s} \cdot \frac{\frac{s}{4}}{\frac{s}{4}}$$

$$T(s) = \frac{1}{0{,}25s + 1}$$

2. $\left.\begin{array}{l} T_{dir} = g \\ T_{loop} = c \cdot g \end{array}\right\} \Rightarrow M(s) = \frac{\frac{4}{0{,}5s+1}}{1 + 4/s} = \frac{4}{(4/s + 1)(0{,}5s+1)} \cdot \frac{\frac{s}{4}}{\frac{s}{4}}$

$$M(s) = \frac{s}{(0{,}5 \cdot s + 1)(0{,}25 \cdot s + 1)}$$

3. $\left.\begin{array}{l} T_{dir} = 1 \\ T_{loop} = c \cdot g \end{array}\right\}$ $S(s) = \frac{1}{1 + 4/s} \cdot \frac{\frac{s}{4}}{\frac{s}{4}}$

$$S(s) = \frac{0{,}25s}{0{,}25s + 1}$$

## 3 PID Controller

1. Two commonly used representations of a PID controller are

   - Series form: $c(s) = K_c \left(\frac{\tau_I s+1}{\tau_I s}\right)(\tau_D s + 1) = \frac{K_c}{\tau_I s}\left(\tau_I \tau_D s^2 + (\tau_I + \tau_D)s + 1\right)$
   - Ideal form: $c(s) = K_c' \left(\frac{1}{\tau_I' s} + \tau_D' s + 1\right) = \frac{K_c'}{\tau_I' s}\left(\tau_I' \tau_D' s^2 + \tau_I' s + 1\right)$

   Given the controller settings $K_c, \tau_I$ and $\tau_D$ for a PID controller in series form, what are the corresponding values of $K_c', \tau_I'$ and $\tau_D'$ for the ideal form? Derive the relationship.

   **Comment:** While one can always derive the parameters for the ideal form from the parameters of the series form, the opposite is not possible. It is not generally possible to go from ideal to series form, because the ideal form can have complex zeros, whereas the series form can only have real zeros $z_1 = -\frac{1}{\tau_I}$ and $z_2 = -\frac{1}{\tau_D}$.

Setting series = Ideal:

$$\frac{K_c}{\tau_I s}\left(\tau_I \tau_D s^2 + (\tau_I + \tau_D)s + 1\right) = \frac{K_c'}{\tau_I' s}\left(\tau_I' \tau_D' s^2 + \tau_I' s + 1\right)$$

$$K_c \tau_D s + \left(K_c + \frac{K_c \tau_D}{\tau_I}\right) + \frac{K_c}{\tau_I s} = K_c' \tau_D' s + K_c' + \frac{K_c'}{\tau_I' s}$$

This means that: $K_c' \tau_D' = K_c \tau_D \Rightarrow \tau_D' = \frac{K_c}{K_c'} \tau_D$

$$K_c' = K_c + \frac{K_c \tau_D}{\tau_I}$$

$$\frac{K_c'}{\tau_I'} = \frac{K_c}{\tau_I} \Rightarrow \tau_I' = \tau_I \cdot \frac{K_c'}{K_c}$$

Immediately: $\underline{\underline{K_c' = K_c\left(1 + \frac{\tau_D}{\tau_I}\right)}}$

Inserting into : $\underline{\underline{\tau_I' = \tau_I + \tau_D}}$

Inserting into : $\tau_D' = \frac{\cancel{K_c}}{\cancel{K_c}\left(1 + \frac{\tau_D}{\tau_I}\right)} \cdot \tau_D = \frac{\tau_D}{1 + \frac{\tau_D}{\tau_I}}$

$$\underline{\underline{\tau_D' = \frac{\tau_D}{\tau_D/\tau_I + 1}}}$$

2. Consider a plant with the transfer function

$$g(s) = \frac{4e^{-3s}}{(10s+1)(4s+1)}. \tag{11}$$

Using the SIMC rules and the rules for model simplification (half-rule):

(a) Derive a first-order with delay approximation to the model $g(s)$ (using the half-rule)

(b) Derive PI controller settings using the SIMC tuning rule (based on the first-order model)

(c) Derive the controller settings for a PID in series (cascade) form

(d) Derive the controller settings for a PID in ideal form

**For a second-order model:**

$$g(s) = \frac{ke^{-\theta s}}{(\tau_1 s+1)(\tau_2 s+1)} \tag{12}$$

**the SIMC tuning rules for PID controller (cascade) are:**

$$K_c = \frac{1}{k}\frac{\tau_1}{\tau_c+\theta} \tag{13}$$

$$\tau_I = \min(\tau_1, 4(\tau_c+\theta)) \tag{14}$$

$$\tau_D = \tau_2 \tag{15}$$

a) $\theta = \theta_0 + \frac{\tau_2}{2} = 3 + \frac{4}{2} = 5$

$\tau_I = \tau_1 + \frac{\tau_2}{2} = 10 + \frac{4}{2} = 12$

$\Rightarrow \underline{\underline{g(s) \approx \frac{4e^{-5s}}{12s+1}}}$

b) Using tight control, $\tau_c = \theta$

$K_c = \frac{1}{k} \cdot \frac{\tau_1}{\tau_c+\theta} = \frac{1}{4} \cdot \frac{12}{2 \cdot 5} = 0{,}3$

$\tau_I = \min(12, 4(5+5)) = 12$

$\Rightarrow \underline{\underline{K_c = 0{,}3,\ \tau_I = 12}}$

c) Using the "exact" transfer function, with $\tau_c = \theta$:

$K_c = \frac{1}{4} \cdot \frac{10}{2 \cdot 3} = \frac{5}{12}$

$\tau_I = \min(10, 4 \cdot (2 \cdot 3)) = 10$

$\tau_D = 4$

$\Rightarrow \underline{\underline{K_c = \frac{5}{12},\ \tau_I = 10,\ \tau_D = 4}}$

d) Using the equations from a):

$K_c' = K_c\left(1 + \frac{\tau_D}{\tau_I}\right) = \frac{5}{12}\left(1 + \frac{4}{10}\right) = \frac{7}{12}$

$\tau_I' = \tau_I + \tau_D = 10 + 4 = 14$

$\tau_D' = \frac{\tau_D}{\tau_D/\tau_I + 1} = \frac{4}{4/10 + 1} = \frac{20}{7}$

$\Rightarrow \underline{\underline{K_c' = \frac{7}{12},\ \tau_I' = 14,\ \tau_D' = \frac{20}{7}}}$