Matte 4N - Øving 7

Oppgave 1

a) IVP: $\begin{cases} y'' - 7y = 0 \\ y(0) = e^{-2} \\ y'(0) = 56 \end{cases}$

2 teorem
- $\mathcal{L}(af+bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$
- $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$

La $\mathcal{L}(y) = Y$

Laplace transformerer: $\mathcal{L}(y'' - 7y) = \mathcal{L}(0) = 0$

$$\mathcal{L}(y''-7y) = s^2 Y - s\cdot y(0) - y'(0) - 7Y = 0$$

$$s^2 Y - se^{-2} - 56 - 7Y = 0$$

$$(s^2-7)Y = se^{-2} + 56$$

$$Y = \frac{se^{-2}+56}{s^2-7}$$

$$Y = \frac{se^{-2}}{s^2-7} + \frac{56}{s^2-7}$$

$$Y = e^{-2}\frac{s}{s^2-7} + \frac{56}{s^2-7}$$

Vil bruke

$\mathcal{L}(\cos h\ at) = \frac{s}{s^2-a^2}$

$\mathcal{L}(\sinh\ at) = \frac{a}{s^2-a^2}$

$$Y = e^{-2}\cdot\frac{s}{s^2-(\sqrt{7})^2} + \frac{56}{\sqrt{7}}\cdot\frac{\sqrt{7}}{s^2-(\sqrt{7})^2}$$

$$y = \mathcal{L}^{-1}(Y) = e^{-2}\mathcal{L}^{-1}\left(\frac{s}{s^2-\sqrt{7}^2}\right) + \frac{56}{\sqrt{7}}\cdot\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2-\sqrt{7}^2}\right)$$

$$\underline{\underline{y = e^{-2}\cosh(\sqrt{7}t) + 8\sqrt{7}\cdot\sin h(\sqrt{7}t)}}$$

b)

$$\text{IVP:}\begin{cases} y''+2y'+y=e^{-2t} \\ y(0)=y'(0)=0 \end{cases}$$

$$\Rightarrow \mathcal{L}(y'')+2\mathcal{L}(y')+\mathcal{L}(y)=\mathcal{L}(e^{-2t}) \quad , \mathcal{L}(y)=Y$$

$$s^2Y-s\cdot y(0)-y'(0)+2(sY-y(0))+Y=\frac{1}{s+2}$$

$$Y(s^2+2s+1)=\frac{1}{s+2}$$

$$Y=\frac{1}{(s+2)(s+1)^2}$$

Bruker delbrøksoppspaltning

$$\frac{1}{(s+2)(s+1)^2}=\frac{A}{(s+2)}+\frac{B}{(s+1)}+\frac{C}{(s+1)^2}$$

$$A(s+1)^2+B(s+2)(s+1)+C(s+2)=1$$

$$As^2+2As+A+Bs^2+3Bs+3B+Cs+2C=1$$

$$\left.\begin{array}{ll} \text{I} & A+B=0 \\ \text{II} & 2A+3B+C=0 \\ & A+2B+2C=1 \end{array}\right\} \Rightarrow \quad \text{I}: A=-B \rightarrow \text{II}: B+C=0 \Rightarrow C=-B \Rightarrow A=C$$

$$\left.\begin{array}{l} A=C \\ A=-B \end{array}\right\} : \text{III} \rightarrow A-2A+2A=1 \Rightarrow \boxed{\begin{array}{l} A=C=1 \\ B=-1 \end{array}}$$

$$\Rightarrow Y=\frac{1}{s+2}-\frac{1}{s+1}+\frac{1}{(s+1)^2}$$

$$y=\mathcal{L}^{-1}(Y)=e^{-2t}-e^{-t}+\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)$$

Her $\mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$, $n=1 \Rightarrow \mathcal{L}(t)=\frac{1}{s^2}$

Ved s-skiftteorem: $\mathcal{L}(e^{at}t)=\frac{1}{(s-a)^2}$, her $a=-1 \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right)=\underline{e^{-t}\cdot t}$

$$\Rightarrow \underline{\underline{y(t)=e^{-2t}-e^{-t}+t\cdot e^{-t}}}$$

c) $y'' + 4y' + 4y = \int_0^t (t-x)^2 \cos(x)\,dx \quad , y(0) = y'(0) = 0$

Viser mindre mellomregninger herfra og ut:

$\int_0^t (t-x)^2 \cos(x)\,dx = \cos t * t^2$ per def

$\mathcal{L}(\cos t * t^2) = \mathcal{L}(\cos t) \cdot \mathcal{L}(t^2) = \frac{\cancel{s}}{s^2+1} \cdot \frac{2}{\cancel{s}} = \underline{\frac{2}{s^2(s^2+1)}}$

$\mathcal{L}(y'' + 4y' + 4y) = Y(s^2+4s+4) = Y(s+2)^2$

$$VS = HS \Rightarrow Y = \frac{2}{s^2(s^2+1)(s+2)^2}$$

Delbrøksoppspaltning

$$\frac{2}{s^2(s^2+1)(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} + \frac{E}{s+2} + \frac{F}{(s+2)^2}$$

$$A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{8}{25},\ D = -\frac{6}{25},\ E = \frac{9}{50},\ F = \frac{1}{10}$$

$$\Rightarrow Y = -\frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s^2} + \frac{8}{25} \cdot \frac{s}{s^2+1} - \frac{6}{25} \cdot \frac{1}{s^2+1} + \frac{9}{50} \cdot \frac{1}{s+2} + \frac{1}{10} \cdot \frac{1}{(s+2)^2}$$

$$y(t) = \mathcal{L}^{-1}(Y) = -\frac{1}{2}u(t) + \frac{t}{2} + \frac{8}{25}\cos(t) - \frac{6}{25}\sin(t) + \frac{9}{50}e^{-2t} + \frac{1}{10}\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)$: Ved s-skift som i c) $\Rightarrow = t \cdot e^{-2t}$

$$\Rightarrow \underline{\underline{y(t) = -\frac{1}{2}u(t) + \frac{t}{2} + \frac{8}{25}\cos(t) - \frac{6}{25}\sin(t) + \frac{9}{50}e^{-2t} + \frac{1}{10}t \cdot e^{-2t}}}$$

d) $y' + y(t) = \delta(t-4) + 1 \qquad y(0) = 2$

$$sY - 2 + Y = e^{-4s} + \frac{1}{s}$$

$$Y(s+1) = e^{-4s} + \frac{1}{s}$$

$$Y = \frac{e^{-4s}}{s+1} + \frac{1}{s(s+1)} + \frac{2}{s+1}$$

$$\boxed{\begin{array}{c} \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \\ \Rightarrow As + A + Bs = 1 \\ A = 1,\ B = -1 \end{array}}$$

$$Y = \frac{e^{-4s}}{s+1} + \frac{1}{s} - \frac{1}{s+1} + \frac{2}{s+1} = \frac{e^{-4s}}{s+1} + \frac{1}{s} + \frac{1}{s+1}$$

$$\underline{\underline{y = \mathcal{L}^{-1}(Y) = u(t-4)e^{4-t} + u(t) + e^{-t}}}$$

$\mathcal{L}^{-1}\left(\frac{e^{-4s}}{s+1}\right)$: 2. skift teorem

$$\mathcal{L}(f(t-a)\cdot u(t-a)) = e^{-as}F(s), \text{ har } a = 4$$

$$F(s) = \frac{1}{s+1} \Rightarrow f(t) = e^{-t}$$

$$\Rightarrow \mathcal{L}^{-1}\left(\frac{e^{-4s}}{s+1}\right) = u(t-4)e^{4-t}$$

$$4 - t = -(t-4)$$

Oppgave 2

a) $$Z(s) = \frac{e^{\pi s}}{e^{\pi}(s-1)} = e^{-\pi} \cdot \frac{e^{\pi s}}{s-1}$$

Byttet bokstav pga. konflikt med teoremet

$$\boxed{\begin{array}{l} t\text{-skift teorem} \\ \text{La } g(t) = f(t-a)u(t-a) \\ \Rightarrow G(s) = e^{-as}F(s) \end{array}}$$

Har $a = -\pi$

$$F(s) = \frac{1}{s-1} \Rightarrow f(t) = e^{t}$$

$$\Rightarrow \mathcal{L}^{-1}(Z(s)) = e^{-\pi}\mathcal{L}^{-1}\left(e^{\pi s}\cdot\frac{1}{s-1}\right) = \cancel{e^{-\pi}} \cdot e^{t+\cancel{\pi}} \cdot u(t+\pi)$$

$$\underline{\underline{\mathcal{L}^{-1}(Z(s)) = e^{t}\cdot u(t+\pi)}}$$

b) Delbrøksoppspaltning

$$Z(s)=\frac{s^2}{(s+2)(s^2+1)^2}=\frac{A}{s+2}+\frac{Bs+C}{s^2+1}+\frac{Ds+E}{(s^2+1)^2}$$

$$\Rightarrow A(s^2+1)^2+(Bs+C)(s+2)(s^2+1)+(Ds+E)(s+2)=s^2$$

$$As^4+2As^2+A+Bs^4+2Bs^3+Bs^2+2Bs+Cs^3+2Cs^2+Cs+2C$$

$$+Ds^2+2Ds+Es+2E=s^2$$

$$\Rightarrow \left.\begin{aligned} s^4&: A+B=0\\ s^3&: 2B+C=0\\ s^2&: 2A+B+2C+D=1\\ s^1&: 2B+C+2D+E=0\\ s^0&: A+2C+2E=0\end{aligned}\right\}\text{Løser:}\quad \begin{aligned} A&=\frac{4}{25}\\ B&=-\frac{4}{25}\\ C&=\frac{8}{25}\\ D&=\frac{1}{5}\\ E&=-\frac{2}{5}\end{aligned}$$

$$\Rightarrow \mathcal{L}^{-1}(Z(s))=\frac{4}{25}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-\frac{4}{25}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)+\frac{8}{25}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)+\frac{1}{5}\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)$$
$$-\frac{2}{5}\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)$$

$$\underline{\mathcal{L}^{-1}(Z(s))=\frac{4}{25}e^{-2t}-\frac{4}{25}\cos(t)+\frac{8}{25}\sin(t)+\frac{1}{5}\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)-\frac{2}{5}\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)=\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right)=\mathcal{L}^{-1}\left(\mathcal{L}(\sin t)\cdot\mathcal{L}(\sin t)\right)=(\sin*\sin)(t)$$

$$=\int_0^t \sin(\tau)\cdot\sin(t-\tau)\,d\tau=\frac{1}{2}\int_0^t \cos(\tau-t+\tau)-\cos(\tau+t-\tau)\,d\tau$$

$$=\frac{1}{2}\int_0^t \cos(2\tau-t)-\cos(t)\,d\tau=\frac{1}{2}\left[\frac{1}{2}\sin(2\tau-t)-\tau\cdot\sin(t)\right]_0^t$$

$$=\frac{1}{2}\left(\frac{1}{2}\sin t-t\cdot\cos t-\left(\frac{1}{2}\sin(-t)-0\right)\right)\qquad /\sin(-t)=-\sin t$$

$$=\frac{1}{2}\left(\frac{1}{2}\sin t+\frac{1}{2}\sin t-t\cdot\cos t\right)=\underline{\frac{1}{2}\left(\sin(t)-t\cdot\cos(t)\right)}$$

Tilsvarende for $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = (\sin * \cos)(t)$

$$= \int_0^t \sin(\tau)\cdot\cos(t-\tau)\,dt = \frac{1}{2}\int_0^t \sin(\tau-t-\tau)+\sin(\tau+t-\tau)\,d\tau$$

$$= \frac{1}{2}\int_0^t \sin(2\tau-t)+\sin(t)\,d\tau = \frac{1}{2}\left[-\frac{1}{2}\cos(2\tau-t)+\tau\cdot\sin t\right]_0^t$$

$$\frac{1}{2}\left(-\frac{1}{2}\cos(t)+t\cdot\sin t-\left(-\frac{1}{2}\cos(-t)+0\right)\right) \quad / \cos(-t)=\cos t$$

$$\underline{\frac{1}{2}t\cdot\sin t}$$

$$\Rightarrow \mathcal{L}^{-1}(Z(s)) = \frac{4}{25}e^{-2t} - \frac{4}{25}\cos(t)+\frac{8}{25}\sin(t)+\frac{1}{10}t\cdot\sin(t)-\frac{1}{5}\sin(t)+\frac{1}{5}t\cdot\cos(t)$$

$$= \frac{4}{25}e^{-2t} - \frac{4}{25}\cos(t)+\frac{3}{25}\sin(t)+\frac{1}{10}t\cdot\sin(t)+\frac{1}{5}t\cdot\cos(t)$$

Felles brøkstrek:

$$\underline{\underline{\mathcal{L}^{-1}(Z(s)) = \frac{8e^{-2t}-8\cos(t)+6\sin(t)+10\,t\cdot\cos(t)+5\,t\cdot\sin(t)}{50}}}$$

Oppgave 3

a) $\mathcal{L}(t^2\cdot\cosh) = \frac{1}{2}\mathcal{L}(t^2e^x+t^2e^{-x})$

$$\boxed{\cosh = \frac{e^t+e^{-t}}{2}}$$

$$= \frac{1}{2}\mathcal{L}(t^2e^t)+\frac{1}{2}\mathcal{L}(t^2e^{-t})$$

[Vi har $f=t^2$, $a=\pm1$]

$$\boxed{\begin{array}{c}\text{s-skift / 1st skift}\\ g(t)=e^{at}f(t)\\ \mathcal{L}(f)=F,\ \mathcal{L}(g)=G\\ \Rightarrow G(s)=F(s-a)\end{array}}$$

$$\mathcal{L}(t^2\cdot\cosh) = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{(s-1)^3}\right)+\frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{(s+1)^3}\right)$$

$$= \frac{1}{(s-1)^3}+\frac{1}{(s+1)^3} = \frac{(s+1)^3+(s-1)^3}{(s-1)^3(s+1)^3}$$

$$\underline{\underline{\mathcal{L}(t^2\cdot\cosh(t)) = \frac{2s^3+6s}{(s^2-1)^3}}}$$

b) $\mathcal{L}(u(t+\frac{\pi}{2})\sin(t))$

Ønsker å bruke t-skift teoremet, men må skrive om $\sin(t)$

$$\boxed{\begin{aligned} &\text{Regel: } \cos(x+\tfrac{\pi}{2}) = -\sin(x) \\ &\Rightarrow \sin(t) = -\cos(t+\tfrac{\pi}{2}) \end{aligned}}$$

$$\boxed{\begin{aligned} &\text{t-skift teorem:} \\ &\mathcal{L}(f(t-a)\cdot u(t-a)) = e^{-as}F(s) \end{aligned}}$$

$$\mathcal{L}(u(t+\tfrac{\pi}{2})\sin(t)) = -\mathcal{L}(u(t+\tfrac{\pi}{2})\cdot\cos(t+\tfrac{\pi}{2}))$$

$$= -e^{\pi/2\, s}\cdot\mathcal{L}(\cos t)$$ (t-skift, $a = -\frac{\pi}{2}$)

$$= -e^{\pi/2\, s}\cdot\frac{s}{s^2+1}$$

$$\mathcal{L}(f(t)) = \frac{-se^{\frac{\pi}{2}s}}{s^2+1}$$

c) $f(t) = \int_0^t \frac{\sin(\pi(t-x))\sqrt{8x}}{t-x}\,dx = \left(\sqrt{8t} * \frac{\sin(\pi t)}{t}\right)(x)$

$$\Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(\sqrt{8t})\cdot\mathcal{L}\left(\frac{\sin(\pi t)}{t}\right)$$

$$\mathcal{L}(\sqrt{8t}) = \sqrt{8}\,\mathcal{L}(\sqrt{t}) = \sqrt{8}\,\frac{\sqrt{\pi}}{2s^{3/2}} = \sqrt{2}\,\frac{\sqrt{\pi}}{s^{3/2}}$$

$$\mathcal{L}\left(\frac{\sin(\pi t)}{t}\right) = ?\qquad \mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\,ds$$

$$\mathcal{L}\left(\frac{\sin(\pi t)}{t}\right) = \int_s^\infty \frac{\pi}{s^2+\pi^2}\,ds = \pi\int_s^\infty \frac{ds}{s^2+\pi^2}\,ds = \pi\left[\frac{1}{\pi}\arctan\left(\frac{s}{\pi}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \arctan\left(\frac{s}{\pi}\right)$$

$$\mathcal{L}(f(t)) = \sqrt{2}\,\frac{\sqrt{\pi}}{s^{3/2}}\left(\frac{\pi}{2} - \arctan\left(\frac{s}{\pi}\right)\right)$$

Tror ikke dette er rett, men aner ikke hvordan man beregner $\mathcal{L}(\sqrt{\pi t})$ og $\mathcal{L}\left(\frac{\sin \pi t}{t}\right)$