Øving 5 - Matematikk 4N

Oppgave 1

Jevn funksjon: $f(x) = f(-x)$

$\hookrightarrow$ Vil vise $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$

- $\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx$

- Substituerer $x = -u$ i: $\int_{-L}^{0} f(x)\,dx = \int_{L}^{0} f(-u)(-du) = \int_0^L f(-u)\,du$

$$\boxed{\begin{array}{c} dx = -du \\ x = -L \Rightarrow u = L \\ x = 0 \Rightarrow u = 0 \end{array}}$$

$$\underset{f(x)=f(-x)}{=} \int_0^L f(u)\,du$$

$$\Rightarrow \int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx = \int_0^L f(u)\,du + \int_0^L f(x)\,dx$$

- nytt variabelskifte $u = x$:

$$= \int_0^L f(x)\,dx + \int_0^L f(x)\,dx$$

$$= 2\int_0^L f(x)\,dx$$

$\Rightarrow \int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$ for jevne funksjoner ▨

Odd funksjon $f(-x) = -f(x)$: Vil vise $\int_{-L}^{L} f(x)\,dx = 0$

- $\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx$

- Samme substitusjon som tidligere: $(x = -u)$

$$\Rightarrow \int_{-L}^{0} f(x)\,dx = \int_0^L f(-u)\,du \underset{f(-x)=-f(x)}{=} -\int_0^L f(u)\,du$$

$$\Rightarrow \int_{-L}^{L} f(x)\,dx = -\int_0^L f(u)\,du + \int_0^L f(x)\,dx \underset{u=x}{=} -\int_0^L f(x)\,dx + \int_0^L f(x)\,dx = \underline{0}$$

$\Rightarrow \int_{-L}^{L} f(x)\,dx = 0$ for odde funksjoner. ▨

BRUNNEN

Oppgave 2 Finner beste approksimasjon av f ved å bruke Gram-Schmidt ortogonalisering til å projisere f ned i V.

I øving 3, oppgave 2 fant vi den ortogonale basen av polynomer for $[0,3]$

$$p_0 = 1, \quad p_1 = x - \frac{3}{2}, \quad p_2 = x^2 - 3x + \frac{3}{2}$$

Projiserer $\cos(x)$ ned i V:

$$f(x) \approx \hat{p}_2(x) = \frac{\langle p_0, f\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p_1, f\rangle}{\langle p_1, p_1\rangle} p_1 + \frac{\langle p_2, f\rangle}{\langle p_2, p_2\rangle} p_2$$

$$= \underline{a p_0 + b p_1 + c p_2}$$

$$a = \frac{\int_0^3 1 \cdot \cos(x)\,dx}{\int_0^3 1^2\,dx} = \underline{\frac{\sin(3)}{3}}$$

$$b = \frac{\int_0^3 (x - \frac{3}{2})\cos(x)\,dx}{\int_0^3 (x-\frac{3}{2})^2\,dx} = \frac{\frac{3}{2}\sin(3) + \cos(3) - 1}{9/4} = \underline{\frac{2}{3}\sin(3) + \frac{4}{9}\cos(3) - \frac{4}{9}}$$

$$c = \frac{\int_0^3 (x^2 - 3x + \frac{3}{2})\cos x\,dx}{\int_0^3 (x^2-3x+\frac{3}{2})^2\,dx} = \frac{-\frac{1}{2}\sin(3) + 3\cos(3) + 3}{27/20}$$

$$\underline{c = -\frac{10}{27}\sin(3) + \frac{20}{9}\cos(3) + \frac{20}{9}}$$

$$\hat{p}_2(x) = a \cdot 1 + b \cdot (x - \tfrac{3}{2}) + c(x^2 - 3x + \tfrac{3}{2})$$

$$= \underline{cx^2 + (b-3c)x + (a - \tfrac{3}{2}b + \tfrac{3}{2}c)} \approx \underline{-0{,}03x^2 - 0{,}70x + 1{,}188}$$

$$\underline{f(x) \approx \hat{p}_2(x) = -0{,}03x^2 - 0{,}70x + 1{,}188}$$

Oppgave 3

a) Kompleks fourierserie: $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ (2π-periodisk)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

For $f(x) = \begin{cases} 3 & -\pi \leq x < 0 \\ 1 & 0 < x \leq \pi \end{cases}$

$$\boxed{\begin{aligned} c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\left([0-(-3\pi)] + [\pi - 0]\right) \\ &= \frac{4\pi}{2\pi} = 2 \Rightarrow \underline{c_0 = 2} \end{aligned}}$$

$$\Rightarrow 2\pi \cdot c_n = \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

$$= \int_{-\pi}^{0} 3e^{-inx}\,dx + \int_{0}^{\pi} e^{-inx}\,dx$$

$$\boxed{e^{-i\pi} = -1}$$

$$= 3\frac{i}{n} e^{-inx}\Big|_{-\pi}^{0} + \frac{i}{n} e^{-inx}\Big|_{0}^{\pi}$$

$$= 3\frac{i}{n}\left[1-(-1)^n\right] + \frac{i}{n}\left[(-1)^n - 1\right]$$

$$= \frac{i}{n}\left(3 - 3(-1)^n + (-1)^n - 1\right)$$

$$= \frac{2i}{n}\left(1-(-1)^n\right)$$

$$\Rightarrow \underline{c_n = \frac{i}{\pi n}\left(1-(-1)^n\right)}$$

$$\boxed{\begin{aligned} &\text{For } n = 2, 4, 6, \ldots \text{ er } c_n = 0 \\ &\phantom{\text{For }} n = 1, 3, 5, \ldots \text{ er } c_n = \frac{2i}{\pi n} \end{aligned}}$$

Omgjør n til oddetall:

$$\Rightarrow \underline{\underline{f(x) \sim \sum_{n=-\infty}^{\infty} \frac{2i}{\pi(2n+1)} e^{i(2n+1)x} + 2}}$$

b) Trigonometrisk fourierrekke: (2π-periodisk)

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$\sin 0 = \sin n\pi = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = c_0 = \underline{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{3}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos(nx)\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{3}{\pi n}\Big[-\cos nx\Big]_{-\pi}^{0} + \frac{1}{\pi n}\Big[-\cos nx\Big]_{0}^{\pi}$$

$$= \frac{3}{\pi n}\left(-1 + (-1)^n\right) + \frac{1}{\pi n}\left(-(-1)^n + 1\right)$$

$$= \underline{\frac{2}{\pi n}\left((-1)^n - 1\right)}$$

$$\boxed{\begin{array}{l} \text{For oddetall er } b_n = \frac{-4}{\pi n} \\ \text{For partall er } b_n = 0 \end{array}}$$

$$\underline{\underline{f(x) \sim 2 - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin((2n+1)x)}{2n+1}}}$$

c) Hvis de samsvarer er $c_0 = a_0$, $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$

$c_0 = a_0 = 2$, ok!

For partall er $c_n = c_{-n} = b_n = 0$, $a_n = 0$

$\Rightarrow$ for partall stemmer de.

Oddetall, $a_n = 0 \Rightarrow c_n = \frac{2i}{\pi n} = -\frac{ib_n}{2} = -\frac{i}{2}\cdot\left(-\frac{4}{\pi n}\right) = \frac{2i}{\pi n}$, ok!

$c_{-n} = -\frac{2i}{\pi n} = \frac{ib_n}{2} = \frac{i}{2}\cdot\left(-\frac{4}{\pi n}\right) = -\frac{2i}{\pi n}$, ok!

$\Rightarrow$ <u>De to fourierseriene stemmer overens</u>

Oppgave 4

a) $f(x) = x^2$ Anta 2π-periodisk, $x \in [-\pi, \pi]$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{\pi^2}{3} \Rightarrow \underline{c_0 = \frac{\pi^2}{3}}$$

$$2\pi\, c_n = \int_{-\pi}^{\pi} \underset{u}{x^2}\, \underset{v'}{e^{-inx}}\,dx = \left[\frac{i}{n} x^2 e^{-inx}\right]_{-\pi}^{\pi} - \frac{2i}{n}\int_{-\pi}^{\pi} \underset{u}{x}\cdot \underset{v'}{e^{-inx}}\,dx$$

$$= \left[\frac{i}{n} x^2 e^{-inx}\right]_{-\pi}^{\pi} - \frac{2i}{n}\cdot\frac{i}{n}\left[x\cdot e^{-inx}\right]_{-\pi}^{\pi} - \frac{2}{n^2}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

$$= \left[\frac{i}{n} x^2 e^{-inx}\right]_{-\pi}^{\pi} + \frac{2}{n^2}\left[x\cdot e^{-inx}\right]_{-\pi}^{\pi} - \frac{2i}{n^3}\left[e^{-inx}\right]_{-\pi}^{\pi}$$

$$= \cancel{\frac{i}{n}\pi^2(-1)^n - \frac{i}{n}\pi^2(-1)^n} + \frac{2}{n^2}\left(\pi(-1)^n + \pi\cdot(-1)^n\right) - \frac{2i}{n^3}\cancel{\left((-1)^n - (-1)^n\right)}$$

$$= \frac{4\pi}{n^2}(-1)^n \Rightarrow c_n = \frac{2(-1)^n}{n^2}$$

$$\Rightarrow \underline{\underline{f(x) \sim \frac{\pi^2}{3} + 2\sum_{n=-\infty, n\neq 0}^{n=\infty} \frac{(-1)^n}{n^2} e^{inx}}}$$

b) $a_0 = c_0 = \frac{\pi^2}{3}$

$$a_n = c_n + c_{-n} = \frac{2(-1)^n}{n^2} + \frac{2(-1)^{-n}}{(-n)^2} = \frac{4(-1)^n}{n^2}$$

$$\boxed{\begin{matrix} n^2 = (-n)^n \\ (-1)^n = (-1)^{-n} \end{matrix}}$$

$$b_n = c_n - c_{-n} = 0$$

$$\Rightarrow \underline{\underline{f(x) \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)}}$$

c) Siden $\int_{-\pi}^{\pi} x^4\,dx$ eksisterer,

Ved parcevals:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$= \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16(-1)^{2n}}{n^4} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16\pi}\int_{-\pi}^{\pi} x^4\,dx - \frac{\pi^4}{72}$$

$$= \frac{1}{16\pi}\cdot\frac{1}{5}(\pi^5+\pi^5) - \frac{\pi^4}{72}$$

$$= \frac{\cancel{2}\pi^{\cancel{5}4}}{\cancel{40}80\cancel{\pi}} - \frac{\pi^4}{72}$$

$$= \frac{\pi^4}{40} - \frac{\pi^4}{72}$$

$$= \underline{\frac{\pi^4}{90}} \quad \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Oppgave 5

a) $f(x) = 3\sin(4x) + 5\cos(3x)$ Antar $x \in [-\pi, \pi]$

$$a_0 = \frac{1}{2\pi}\int 3\sin(4x) + 5\cos(3x)\,dx = 0$$

Vet at $\int_{-\pi}^{\pi} \sin(nx)dx = 0$, $\int_{-\pi}^{\pi}\cos(nx) = 0$ for alle $n \in \mathbb{Z}$ uten om 0

$$\pi a_n = 3\underbrace{\int_{-\pi}^{\pi} \sin(4x)\cdot\cos(nx)dx}_{\substack{\text{odde funksjon} \\ \Rightarrow = 0}} + 5\int_{-\pi}^{\pi}\cos(3x)\cos(nx)\,dx$$

$$\pi a_n = 5\int_{-\pi}^{\pi}\cos(3x)\cos(nx)\,dx$$

$$= 5\int_{-\pi}^{\pi}\frac{1}{2}\left(\cos((3+n)x) + \cos((3-n)x)\right)dx$$

$\Rightarrow \pi a_n = 0$ dersom $(3+n), (3-n) \in \mathbb{Z}$ og $\neq 0$

Siden $n \geq 1$, vil kun $3-n = 0$ når $n = 3$

$$\Rightarrow \pi a_3 = 5\int_{-\pi}^{\pi} \frac{1}{2}\left(\cos(6x)+\cos(0)\right)dx$$

$$= \frac{5}{2}2\pi$$

$$\underline{a_3 = 5}$$

Siden $a_n = 0$ for alle $n \neq 3$, vil $\sum_{n=1}^{\infty} a_n \cos(nx) = 5\cos(3x)$

Tilsvarende

$$\pi b_n = 3\int_{-\pi}^{\pi} \sin(4x)\sin(nx)dx + \underbrace{5\int_{-\pi}^{\pi} \cos(3x)\sin(nx)\,dx}_{=0,\text{ odde funksjon}}$$

$$= 3\int_{-\pi}^{\pi} \sin(4x)\sin(nx)dx$$

$$= \frac{3}{2}\int_{-\pi}^{\pi} \cos((4-n)x) - \cos((4+n)x)dx$$

$\Rightarrow \pi b_n = 0$ når $n \neq 4$, for $n=4$:

$$\pi b_4 = \frac{3}{2}\int_{-\pi}^{\pi} \cos(0) - \cos(8x)\,dx$$

$$\pi b_4 = \frac{3}{2}2\pi$$

$$\underline{b_4 = 3}$$

Siden $b_n = 0$ for alle $n \neq 4$, vil $\sum_{n=1}^{\infty} b_n \sin(nx) = 3\sin(4x)$

$$\Rightarrow \underline{\underline{f(x) \sim 3\sin(4x) + 5\cos(3x) = f(x)}}$$

Det er som forventet siden f allerede lå i rommet

til de trigonometriske funksjonene.

Oppgave 6

La $h(x) = x$

$$\Rightarrow g(x) = h(x) + f(x)$$

I forelesningene fant vi at $h(x) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

I oppgave 3 fant vi $f(x) \sim 2 + \sum_{n=1}^{\infty} \frac{2}{\pi n}\left((-1)^n - 1\right) \sin nx$

$$\Rightarrow g(x) \sim 2 + \sum_{n=1}^{\infty} \frac{2}{\pi n}\left((-1)^n - 1\right) \sin nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$= 2 + 2\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n} + \frac{\pi(-1)^{n+1}}{\pi n}\right) \sin nx$$

$$= 2 + 2\sum_{n=1}^{\infty} \frac{(-1)^n + \pi(-1)^{n+1} - 1}{\pi n} \sin nx$$

$$= 2 + 2\sum_{n=1}^{\infty} \frac{(-1)^n - \pi(-1)^n - 1}{\pi n} \sin nx$$

$$= 2 + 2\sum_{n=1}^{\infty} \frac{(-1)^n(1-\pi) - 1}{\pi n} \sin nx$$

$$\underline{\underline{g(x) \sim 2 + 2\sum_{n=1}^{\infty} \frac{(-1)^n(1-\pi) - 1}{\pi n} \sin(nx)}}$$