Øving 4 - Matematikk 4N

Oppgave 1

a) På intervallet $[0,1]$, er $0 \leq f(x) \leq 1 \Rightarrow$ $\underline{\underline{f \text{ har minst ett fikspunkt } x^*}}$

Hvis $f \in C^1(0,1)$ og $|f'(x)| \leq C < 1$ på $x \in [0,1]$, er x^* unikt.

$$f'(x) = x - \frac{1}{2}\sin(x), \Rightarrow f \in C^1(0,1)$$

Finner maksverdi:

$$f''(x) = 0$$

$1 - \frac{1}{2}\cos(x) = 0$, ingen løsning, men $1 - \frac{1}{2}\cos(x) > 0$ for $x \in [0,1]$

$\Rightarrow f'(x)$ er stigende for $x \in [0,1]$

$$C = \text{maks } f'(x) = f(1) = 1 - \frac{1}{2}\sin(1) \approx 0{,}58 < 1 \Rightarrow |f'(x)| \leq C < 1,\ x \in [0,1]$$

$\underline{\underline{\text{Siden } f \in C^1(a,b) \text{ og } |f'(x)| < 1\ \forall x \in [0,1] \text{, finnes en unik}}}$

$\underline{\underline{x^* \text{ på } x \in [0,1]}}$ $\square$

b) $x^{(1)} = f(x^{(0)}) = f(0) = 0 + \frac{1}{2}\cos(0) = \frac{1}{2}$

$$\underline{\underline{x^{(1)} = \frac{1}{2}}}$$

c) $|x^{(k)} - x^*| \leq \dfrac{C^k}{1-C}|x^{(1)} - x^{(0)}| \leq 10^{-6}$

$$\frac{0{,}58^k}{1-0{,}58} \cdot \frac{1}{2} \leq 10^{-6}$$

$$k \cdot \ln(0{,}58) \leq \ln(2 \cdot 0{,}42 \cdot 10^{-6})$$

$$k \geq \frac{\ln(2 \cdot 0{,}42 \cdot 10^{-6})}{\ln(0{,}58)} = 25{,}68$$

$$\underline{\underline{k \geq 26}}$$

BRUNNEN

Oppgave 2 Banach teoremet: $\frac{1}{4}\sin(x_1+x_2)$ og $\frac{1}{4}\cos(x_1-x_2)$

a) • har kontinuerlige førstederiverte i $\mathbb{R}^2 (\in \mathbb{R}^2)$. $F(x) \in \mathbb{R}^2 \ \forall x \in \mathbb{R}^2$

$$J_F(x) = \begin{pmatrix} \frac{1}{4}\cos(x_1+x_2) & \frac{1}{4}\cos(x_1+x_2) \\ -\frac{1}{4}\sin(x_1-x_2) & \frac{1}{4}\sin(x_1-x_2) \end{pmatrix}$$

La $\|\cdot\|$ være definert ved:

$\|\cdot\| = \|\cdot\|_1$, hvor for en matrise A; $\|A\|_1 = \max_{1\leq j\leq 2} \sum_{i=1}^{2} |a_{ij}|$

hvor a_{ij} er komponenten i rad i og kolonne j.

Etbersom $\sin(x)$ og $\cos(x) \leq 1 \ \forall x \in \mathbb{R}$, ved triangelulikheten er:

• $\|J_F(x)\| \leq \frac{1}{2} < 1 \ \forall x \in \mathbb{R}^2$. $\left(C = \frac{1}{2}\right)$

• Ved Banach fikspunktteorem finnes det ett unikt fikspunkt i $\mathbb{R}^2$ $\square$

b) $x^{(1)} = F(x^{(0)}) = F\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix} \frac{1}{4}\sin(0)+1 \\ \frac{1}{4}\cos(0)+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 9/4 \end{pmatrix}$

$$\underline{\underline{x^{(1)} = \begin{pmatrix} 1 \\ 9/4 \end{pmatrix}}}$$

c) Feilen er begrenset ved:

$$\|e(x^{(k)})\|_1 = \|x^{(k)} - x^*\|_1 \leq \frac{C^k}{1-C} \|x^{(1)} - x^{(0)}\|_1$$

$C = \frac{1}{2}$, $x^{(0)} = \begin{pmatrix}0\\0\end{pmatrix}$, $x^{(1)} = \begin{pmatrix}1\\9/4\end{pmatrix}$

$$\|e(x^{(k)})\|_1 \leq \frac{(1/2)^k}{1/2} \left\|\begin{pmatrix}1\\9/4\end{pmatrix}\right\|_1 \leq 10^{-6}$$

$$\frac{1}{2^{k-1}} \cdot \frac{13}{4} = \frac{13}{2^{k+1}} \leq 10^{-6}$$

$$\frac{2^{k+1}}{13} \geq 10^6 \Rightarrow 2^{k+1} \geq 13 \cdot 10^6$$

$(k+1) \cdot \ln 2 \geq \ln(13 \cdot 10^6) \Rightarrow k \geq \frac{\ln(13\cdot 10^6)}{\ln(2)} - 1 = 22{,}63\ldots$

$\underline{\underline{k \geq 23 \Rightarrow \text{Må iterere 23 ganger for å få feil} \leq 10^{-6} \text{ (garantert} \leq 10^{-6})}}$

Oppgave 3

Newtons metode:

$$\boxed{x^{(k+1)} = x^{(k)} - J_F^{-1}(x^{(k)}) \cdot F(x^{(k)})}$$

$$J_F(x) = \begin{pmatrix} 2x_1 & -2x_2 \\ 2 & -1 \end{pmatrix}$$

$$J_F^{-1}(x) = \frac{1}{\det(J_F(x))} \begin{pmatrix} -1 & 2x_2 \\ -2 & 2x_1 \end{pmatrix} = \frac{1}{-2x_1 - (-2x_2)} \begin{pmatrix} -1 & 2x_2 \\ -2 & 2x_1 \end{pmatrix}$$

$$= \frac{1}{2x_2 - 2x_1} \begin{pmatrix} -1 & 2x_2 \\ -2 & 2x_1 \end{pmatrix}$$

$$J_F^{-1}\left(\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right) = \frac{1}{2 \cdot 2 - 2 \cdot \frac{1}{2}} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ -2/3 & 1/3 \end{pmatrix}$$

- $$x^{(1)} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/3 & 2/3 \\ -2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/12 \\ -1/6 \end{pmatrix} = \begin{pmatrix} 7/12 \\ 7/6 \end{pmatrix}$$

$$\underline{\underline{x^{(1)} = \left(\frac{7}{12}, \frac{7}{6}\right)^T}}$$

BRUNNEN