exercises_4

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1 Exercises 4: Nonlinear equations

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If you want to have a nicer theme for your jupyter notebook, download the cascade stylesheet file tma4125.css and execute the next cell:

[1]:
```
from IPython.core.display import HTML
def css_styling():
    try:
        with open("tma4125.css", "r") as f:
            styles = f.read()
            return HTML(styles)
    except FileNotFoundError:
        pass #Do nothing

# Comment out next line and execute this cell to restore the default notebook␣
↪style
css_styling()
```

[1]:
```
<IPython.core.display.HTML object>
```

1.0.1 Problem 1

Consider the equation $x = f(x)$ where $f(x) = \frac{1}{2}x^2 + \frac{1}{2}\cos(x)$.

a) Show that f has a unique fixed point $x^\star$ in $[0, 1]$.

Solution:

Øving 4 - Matematikk 4N

Oppgave 1

a) På intervallet $[0,1]$, er $0 \leq f(x) \leq 1 \Rightarrow f$ har minst ett fikspunkt x^*

Hvis $f \in C^1(0,1)$ og $|f'(x)| \leq C < 1$ på $x \in [0,1]$, er x^* unikt.

$f'(x) = x - \frac{1}{2}\sin(x), \Rightarrow f \in C^1(0,1)$

Finner maksverdi:

$f''(x) = 0$

$1 - \frac{1}{2}\cos(x) = 0$, ingen løsning, men $1 - \frac{1}{2}\cos(x) > 0$ for $x \in [0,1]$

$\Rightarrow f'(x)$ er stigende for $x \in [0,1]$

$C = \text{maks } f'(x) = f'(1) = 1 - \frac{1}{2}\sin(1) \approx 0{,}58 < 1 \Rightarrow |f'(x)| \leq C < 1,\ x \in [0,1]$

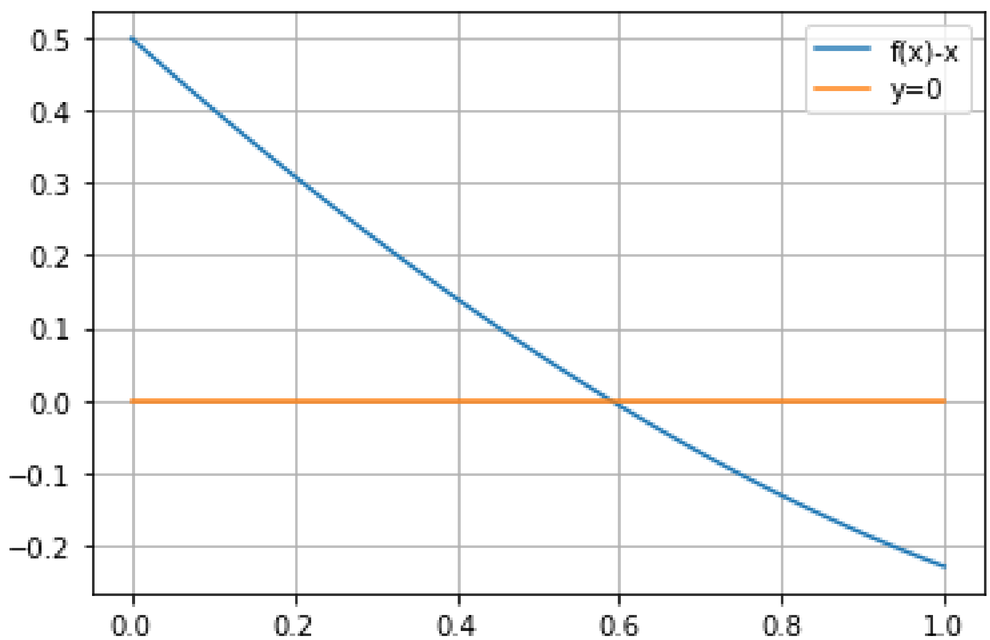
Siden $f \in C^1(a,b)$ og $|f'(x)| < 1\ \forall x \in [0,1]$, finnes en unik x^* på $x \in [0,1]$ $\square$

[2]:
```python
#Plotter den og
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return 0.5*x**2 + 0.5*np.cos(x)

a, b = 0, 1
x = np.linspace(a,b,101)

plt.plot(x,f(x)-x, label = "f(x)-x")
plt.plot(x,x-x, label = "y=0")
plt.legend()
plt.grid(True)
```



b) Take one step with the Fixed-point iteration method and $x^{(0)} = 0$. What is $x^{(1)}$?

Solution:

b) $x^{(1)} = f(x^{(0)}) = f(0) = 0 + \frac{1}{2}\cos(0) = \frac{1}{2}$

$$\underline{\underline{x^{(1)} = \frac{1}{2}}}$$

c) Estimate how many iterations you have to perform to ensure that the error $|x^{(k)} - x^{\star}| \leq 10^{-6}$.

Solution:

c) $|x^{(k)} - x^*| \leq \frac{C^k}{1-C} |x^{(1)} - x^{(0)}| \leq 10^{-6}$

$$\frac{0{,}58^k}{1-0{,}58} \cdot \frac{1}{2} \leq 10^{-6}$$

$$k \cdot \ln(0{,}58) \leq \ln(2 \cdot 0{,}42 \cdot 10^{-6})$$

$$k \geq \frac{\ln(2 \cdot 0{,}42 \cdot 10^{-6})}{\ln(0{,}58)} = 25{,}68$$

$$\underline{\underline{k \geq 26}}$$

1.0.2 Problem 2

Recall that for systems of equations $\mathbf{x} = F(\mathbf{x})$ the Banach Fixed-point theorem says that if there is a closed subset $D \subseteq \mathbb{R}^2$ (note that we can have $D = \mathbb{R}^2$) such that $F(\mathbf{x}) \in D$ for every $\mathbf{x} \in D$ and $F \in C^1(D)$ with $\|J_F(\mathbf{x})\| \leq C < 1$ for every $\mathbf{x} \in D$ then

1. F has a unique fixed point $\mathbf{x}^\star$ in D.
2. The Fixed-point iteration converges linearly to $\mathbf{x}^\star$ for every $\mathbf{x}^{(0)} \in D$.
3. The error after k iterations is bounded by

$$\|x^{(k)} - x^\star\| \leq \tfrac{C^k}{1-C}\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\|.$$

Note: Here $\|\cdot\|$ is any norm (for example $\|\cdot\|_1$) and J_F is measured in the corresponding matrix-norm (for example $\|\cdot\|_1$).

Consider the equation $\mathbf{x} = F(\mathbf{x})$ where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$F(\mathbf{x}) = \begin{pmatrix} \frac{1}{4}\sin(x_1 + x_2) \\ \frac{1}{4}\cos(x_1 - x_2) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Show that F has a unique fixed point $\mathbf{x}^\star$ in $\mathbb{R}^2$.

Solution:

Oppgave 2 Banach teoremet: $\frac{1}{4}\sin(x_1+x_2)$ og $\frac{1}{4}\cos(x_1-x_2)$

a) • har kontinuerlige førstederiverte i $\mathbb{R}^2 (\subseteq \mathbb{R}^2)$. $F(x) \in \mathbb{R}^2 \ \forall x \in \mathbb{R}^2$

$$J_F(x) = \begin{pmatrix} \frac{1}{4}\cos(x_1+x_2) & \frac{1}{4}\cos(x_1+x_2) \\ -\frac{1}{4}\sin(x_1-x_2) & \frac{1}{4}\sin(x_1-x_2) \end{pmatrix}$$

La $\|\cdot\|$ være definert ved:

$\|\cdot\| = \|\cdot\|_1$, hvor for en matrise A: $\|A\|_1 = \max_{1\le j\le 2} \sum_{i=1}^{2} |a_{ij}|$

hvor a_{ij} er komponenten i rad i og kolonne j.

Ettersom $\sin(x)$ og $\cos(x) \leq 1 \ \forall x \in \mathbb{R}$, ved trekantulikheten er:

• $\|J_F(x)\| \leq \frac{1}{2} < 1 \ \forall x \in \mathbb{R}^2$. $(C = \frac{1}{2})$

• Ved Banach fikspunktteorem finnes det ett unikt fikspunkt i $\mathbb{R}^2$ □

b) Take one step with the Fixed-point iteration method and $\mathbf{x}^{(0)} = (0,0)^T$. What is $\mathbf{x}^{(1)}$?

Solution:

b) $x^{(1)} = F(x^{(0)}) = F\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{4}\sin(0)+1\\ \frac{1}{4}\cos(0)+2\end{pmatrix} = \begin{pmatrix}1\\9/4\end{pmatrix}$

$\underline{\underline{x^{(1)} = \begin{pmatrix}1\\9/4\end{pmatrix}}}$

c) Estimate how many iterations you have to perform to ensure that the error $\|\mathbf{x}^{(k)} - \mathbf{x}^{\star}\|_1 \leq 10^{-6}$.

Solution:

c) Feilen er begrenset ved:

$$\|e(x^{(k)})\|_1 = \|x^{(k)} - x^{*}\|_1 \leq \frac{C^k}{1-C}\|x^{(1)} - x^{(0)}\|_1$$

$$C = \frac{1}{2},\ x^{(0)} = \begin{pmatrix}0\\0\end{pmatrix},\ x^{(1)} = \begin{pmatrix}1\\9/4\end{pmatrix}$$

$$\|e(x^{(k)})\|_1 \leq \frac{(1/2)^k}{1/2}\left\|\begin{pmatrix}1\\9/4\end{pmatrix}\right\|_1 \leq 10^{-6}$$

$$\frac{1}{2^{k+1}} \cdot \frac{13}{4} = \frac{13}{2^{k+1}} \leq 10^{-6}$$

$$\frac{2^{k+1}}{13} \geq 10^{6} \Rightarrow 2^{k+1} \geq 13 \cdot 10^{6}$$

$$(k+1) \cdot \ln 2 \geq \ln(13 \cdot 10^{6}) \Rightarrow k \geq \frac{\ln(13 \cdot 10^{6})}{\ln(2)} - 1 = 22{,}63\ldots$$

$\underline{\underline{k \geq 23 \Rightarrow \text{Må iterere 23 ganger for å få feil} \leq 10^{-6}}}$ (garantert $\leq 10^{-6}$)

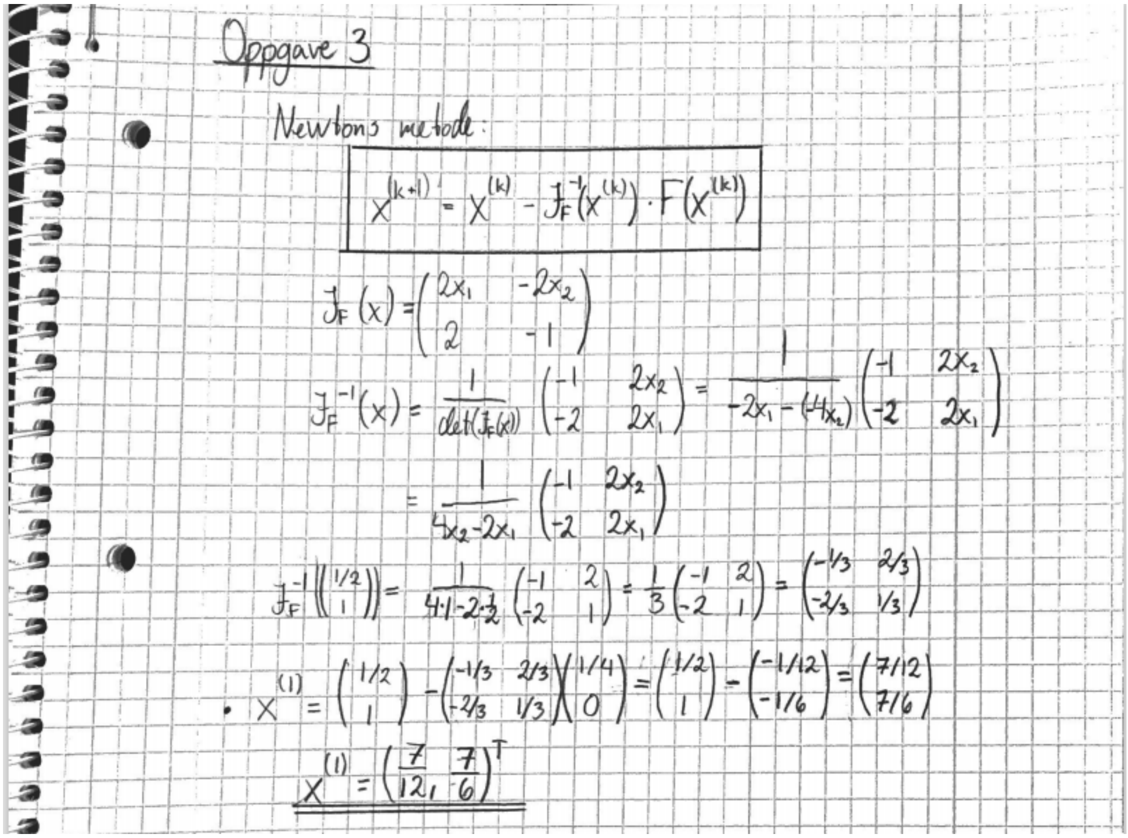
1.0.3 Problem 3

Consider the equation $F(\mathbf{x}) = 0$ where $F : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$F(\mathbf{x}) = \begin{pmatrix}1 + x_1^2 - x_2^2\\ 2x_1 - x_2.\end{pmatrix}$$

Take one step with Newton's method and $\mathbf{x}^{(0)} = (1/2, 1)^T$. What is $\mathbf{x}^{(1)}$?

Solution:



Oppgave 3

Newtons metode:

$$\boxed{X^{(k+1)} = X^{(k)} - J_F^{-1}(X^{(k)}) \cdot F(X^{(k)})}$$

$$J_F(x) = \begin{pmatrix} 2x_1 & -2x_2 \\ 2 & -1 \end{pmatrix}$$

$$J_F^{-1}(x) = \frac{1}{\det(J_F(x))} \begin{pmatrix} -1 & 2x_2 \\ -2 & 2x_1 \end{pmatrix} = \frac{1}{-2x_1 - (-4x_2)} \begin{pmatrix} -1 & 2x_2 \\ -2 & 2x_1 \end{pmatrix}$$

$$= \frac{1}{4x_2 - 2x_1} \begin{pmatrix} -1 & 2x_2 \\ -2 & 2x_1 \end{pmatrix}$$

$$J_F^{-1}\left(\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}\right) = \frac{1}{4 \cdot 1 - 2 \cdot \frac{1}{2}} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ -2/3 & 1/3 \end{pmatrix}$$

$$\bullet\ X^{(1)} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/3 & 2/3 \\ -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/12 \\ -1/6 \end{pmatrix} = \begin{pmatrix} 7/12 \\ 7/6 \end{pmatrix}$$

$$X^{(1)} = \left(\frac{7}{12}, \frac{7}{6}\right)^T$$

1.0.4 Problem 4

We will now attempt to solve a non-linear equation numerically. Consider the equation $f(x) = 0$ where $f(x) = \arctan x + 1$.

a) What is the root $x^\star$ of f?

Solution:

$$f(x) = 0$$
$$\arctan x + 1 = 0$$
$$\arctan x = -1$$
$$x = \tan(-1)$$
$$x \approx -1.5574$$

b) Implement Newton's method for the function f in Python. The function should iterate until $|f(x^{(k)}| \leq$ `tol`. You should be able to change the tolerance `tol` and the initial guess $x^{(0)}$. If it converges, what is the error $|x^{(k)} - x^\star|$ after each iteration?

Hint: It is recommended that you ensure that the program iterates at most some large number of times, say 10000*. There is a skeleton of a code provided below.*

[3]:
```python
import numpy as np
from math import atan

def f(x):
    return atan(x) + 1

def df(x):
    return 1/((x**2) + 1)

answer = np.tan(-1)

def newton(x):
    x_new = x - ((f(x))/(df(x)))
    return x_new

def newton_iter(x, tol):
    print(f"x0 = {x}:")
    k = 0
    x0 = x
    error = abs(x - newton(x))
    while error>tol and k<10000:
        k+=1
        try:
        #Tar ett newton steg
            x = newton(x)
            error = abs(x - answer)
            print(f"x* = {x}, med feil {error:.5e}. Antall iterasjoner: {k}")
        except OverflowError:
        #Stopper iterasjonen hvis det blir for store tall
            error = False
            break
    if error == False:
        print(f"Med x0 = {x0} blir det ikke konvergerens")
    else:
        print(f"Iterasjonen endte etter {k} iterasjoner, x* = {x}")
```

c) Try your method with tolerance `tol` $= 10^{-6}$ and the two initial guesses $x^{(0)} = 1.2, 1.4$. Does the iteration converge for both initial guesses?

Solution:

[4]:
```python
newton_iter(1.2 ,10**(-6))

print(2*'\n')
newton_iter(1.4 ,10**(-6))
```

```
x0 = 1.2:
x* = -3.3775816434595916, med feil 1.82017e+00. Antall iterasjoner: 1
```

```
x* = 0.13326698195898112, med feil 1.69067e+00. Antall iterasjoner: 2
x* = -1.0193324149621643, med feil 5.38075e-01. Antall iterasjoner: 3
x* = -1.4373933586891008, med feil 1.20014e-01. Antall iterasjoner: 4
x* = -1.5509630650674628, med feil 6.44466e-03. Antall iterasjoner: 5
x* = -1.5573888566923006, med feil 1.88680e-05. Antall iterasjoner: 6
x* = -1.5574077244930478, med feil 1.61855e-10. Antall iterasjoner: 7
Iterasjonen endte etter 7 iterasjoner, x* = -1.5574077244930478
```

```
x0 = 1.4:
x* = -4.373618648803744, med feil 2.81621e+00. Antall iterasjoner: 1
x* = 2.591189397822207, med feil 4.14860e+00. Antall iterasjoner: 2
x* = -14.399348352914659, med feil 1.28419e+01. Antall iterasjoner: 3
x* = 90.07545997958847, med feil 9.16329e+01. Antall iterasjoner: 4
x* = -20680.79596393944, med feil 2.06792e+04. Antall iterasjoner: 5
x* = 244085557.59375292, med feil 2.44086e+08. Antall iterasjoner: 6
x* = -1.531622846024825e+17, med feil 1.53162e+17. Antall iterasjoner: 7
x* = 1.3390131471828251e+34, med feil 1.33901e+34. Antall iterasjoner: 8
x* = -4.609325234474894e+68, med feil 4.60933e+68. Antall iterasjoner: 9
x* = 1.2127069759607346e+137, med feil 1.21271e+137. Antall iterasjoner: 10
x* = -3.780762723066036e+274, med feil 3.78076e+274. Antall iterasjoner: 11
Med x0 = 1.4 blir det ikke konvergerens
```

Itereringen konvergerer ikke for $x^{(0)} = 1.4$

d) *Optional*

The root $x^\star$ of f is a fixed point of the function $g(x) = x - \arctan(x) - 1$. Implement the Fixed-point method for the function g, with the ability to change $x^{(0)}$ and `tol`.

Does the iteration converge for `tol` $= 10^{-6}$ and the two inital guesses $x^{(0)} = 1.2, 1.4$? Feel free to test other values of $x^{(0)}$.

If it converges, what is the error $|x^{(k)} - x^\star|$ after each iteration?

How many iterations does it take to converge? Compare this to the number of iteration it takes for Newton's method to converge (if it converges).

Solution:

Prebuilt skeleton of a code

[5]:
```
import numpy as np
from math import atan

def newton(x):
    #Should return the next iterate after one Newton step.
    return x_new

def newton_iter(x, tol):
```

```
    while error>tol and iterations<10000:

        try:
            s = False
            #Take one newton step
        except OverflowError: #This prints out a message if the numbers are too␣
↪large to handle.
            print("Did not converge.")
            break
```