Matematikk 4N - Øving 11

Oppgave 1

a) Gitt i (4): $F_n(x)G_n(t) = \left(A_n \cos\left(c\frac{n\pi}{L}t\right) + B_n \sin\left(c\frac{n\pi}{L}t\right)\right)\cos\left(\frac{n\pi}{L}x\right)$

for $n = 0, 1, 2, 3, \ldots$

Frekvens er gitt med $f_n = \frac{\omega_n}{2\pi}$, $\omega_n = c\frac{n\pi}{L}$

$$\Rightarrow f_n = \frac{cn}{2L}$$, ser at $n=1$ gir lavest frekvens ($n=0$ gir ingen lyd)

$$f_1 = \frac{343 \cdot 1}{2 \cdot 0{,}70} = 245\ s^{-1}$$

$$\underline{\underline{\text{Lavest frekvens er } 245\ s^{-1}}}$$

b) Ved separasjon av variable: $u(x,t) = F(x)G(t)$

$$\Rightarrow F(x)G''(t) = c^2 F''(x)G(t)$$

$$\frac{G''(t)}{G(t)} = c^2 \frac{F''(x)}{F(x)} = k$$

• Løser for F: $F''(x) = \frac{k}{c^2}F(x)$

Dersom $\frac{k}{c^2} = 0$, $F''(x) = 0 \Rightarrow F(x) = ax + b$

$$F'(x) = a$$

Ettersom $F'(0) = F'(L) = 0 \Rightarrow a = 0$

$$\Rightarrow \underline{F(x) = b}$$

- Dersom $\frac{k}{c^2} = -\rho^2 < 0$

$$F''(x) = -\rho^2 F(x) \qquad \rho > 0$$

$$\Rightarrow F(x) = A\sin(\rho x) + B\cos(\rho x)$$

$$F'(x) = A\rho\cos(\rho x) - B\rho\sin(\rho x)$$

B.C $F'(0) = F'(L) = 0$

$$F'(0) = A\rho = 0 \Rightarrow \underline{A = 0}$$

$$F'(L) = -B\rho\sin(\rho L) = 0 \Rightarrow \rho = \frac{n\pi}{L} \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow F_n(x) = B\cos\left(\frac{n\pi}{L}x\right) \quad n = 1, 2, \ldots$$

Ved å la $n=0$, vil løsningen fra $\frac{k}{c^2} = 0$ inkluderes

$$\Rightarrow F_n(x) = B\cos\left(\frac{n\pi}{L}x\right) \quad n = 0, 1, 2, \ldots$$

- Dersom $\frac{k}{c^2} = \rho^2 > 0$

$$F''(x) = \rho^2 F(x)$$

$$\Rightarrow F(x) = Ae^{\rho x} + Be^{-\rho x}$$

$$F'(x) = A\rho e^{\rho x} - B\rho e^{\rho x}$$

- B.C: $F'(0) = 0 \Rightarrow A\rho - B\rho = 0 \Rightarrow A = B$

$$\Rightarrow F'(x) = A\rho\left(e^{\rho x} - e^{-\rho x}\right)$$

- B.C: $F'(L) = 0$

$$F'(L) = A\rho\left(e^{\rho L} - e^{-\rho L}\right) = 0$$

Ettersom $\rho > 0$, vil $e^{\rho L} > e^{-\rho L} \Rightarrow A = 0$

$\Rightarrow$ Trivell løsning

Får at alle $F_n(x)$ er gitt ved

$$F_n(x) = B_n \cos\left(\frac{n\pi}{L} x\right) \quad n = 0,1,2,\dots$$

• Løser for G, Ettersom $\frac{k}{c^2} = -p^2 = -\left(\frac{n\pi}{L}\right)^2 \Rightarrow k = -\left(\frac{cn\pi}{L}\right)^2$

$$G''(t) = k\,G(t) = -\left(\frac{cn\pi}{L}\right)^2 G(t)$$

$$\Rightarrow G_n(t) = C_n \sin\left(\frac{cn\pi}{L} t\right) + D_n \cos\left(\frac{cn\pi}{L} t\right) \quad n = 0,1,2,3,\dots$$

Ved å kombinere $F_n(x) G_n(t)$, og ved å samle konstantene:

$$\underline{\underline{u(x,t) = \sum_{n=0}^{\infty} \left(A_n \cos\left(c\frac{n\pi}{L} t\right) + B_n \sin\left(c\frac{n\pi}{L} t\right)\right) \cos\left(\frac{n\pi}{L} x\right)}}$$

– I.C. $f(x) = u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right)$

Dette er en cosinus-Fourierrekke, vet da at A_n er gitt ved

$$\underline{\underline{A_0 = \frac{1}{L}\int_0^L f(x)\,dx}}$$

$$\underline{\underline{A_n = \frac{2}{L}\int f(x) \cos\left(\frac{n\pi}{L} x\right) dx \quad n = 1,2,3,\dots}}$$

$$u_t(x,t) = \sum_{n=0}^{\infty} \left(-A_n c\frac{n\pi}{L} \sin\left(\frac{cn\pi}{L} t\right) + B_n c\frac{n\pi}{L} \cos\left(c\frac{n\pi}{L} x\right)\right) \cos\left(\frac{n\pi}{L} x\right)$$

– I.C. $g(x) = u_t(x,0) = \sum_{n=0}^{\infty} B_n c\frac{n\pi}{L} \cos\left(\frac{n\pi}{L} x\right)$

Tilsvarende: $B_n c\frac{n\pi}{L} = \frac{2}{L}\int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx$

$$\underline{\underline{\Rightarrow B_n = \frac{2}{cn\pi}\int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx \quad n = 1,2,3,\dots}}$$

$n = 0 \Rightarrow$ at hele leddet forsvinner, den trengs dermed ikke

$(B_0 \sin(0) = 0 \ \forall\ B_0)$

Oppgave 2

Teorem 4.1 med $c=4$, $L=1$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(4n\pi t) + B_n \sin(4n\pi t)\right)\sin(n\pi x)$$

$$\text{med } A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$B_n = \frac{1}{2n\pi}\int_0^1 g(x)\sin(n\pi x)\,dx$$

$$A_n: \quad f(x) = 1-|2x-1| = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2(1-x), & 1/2 < x \leq 1 \end{cases}$$

$$\Rightarrow A_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = \int_0^{1/2} 4x\sin(n\pi x)\,dx + \int_{1/2}^1 4(1-x)\sin(n\pi x)\,dx$$

Delvis integrasjon:

$$A_n = \underbrace{\left[-4x \cdot \frac{\cos(n\pi x)}{n\pi}\right]_0^{1/2}}_{=0} + \frac{4}{n\pi}\int_0^{1/2}\cos(n\pi x)\,dx + \underbrace{\left[-4(1-x)\frac{\cos(n\pi x)}{n\pi}\right]_{1/2}^1}_{=0} - \frac{4}{n\pi}\int_{1/2}^1 \cos(n\pi x)\,dx$$

$$= \frac{4}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^{1/2} - \frac{4}{n^2\pi^2}\left[\sin(n\pi x)\right]_{1/2}^1$$

$$= \frac{4}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi) + \sin\frac{n\pi}{2}\right)$$

$$= \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$ (Her er det noe rart i LF)

$$= \begin{cases} 4/n^2\pi^2 & n=4k+1 \\ -4/n^2\pi^2 & n=4k+3 \\ 0 & \text{, ellers} \end{cases}$$ $\Leftarrow$ Tror det skal være $\frac{8}{n^2\pi^2}$

$$\Rightarrow A_n \text{ kan skrives som } A_{4k+2j+1} = \frac{4(-1)^j}{(4k+2j+1)^2\pi^2} \quad j=0,1,\ k=0,1,2,\ldots$$

$$B_n = \frac{1}{2n\pi}\int_0^1 \sin(n\pi x)\, x(1-x)\,dx$$

$$= \frac{1}{2n\pi}\underbrace{\left[-\frac{\cos(n\pi x)}{n\pi}x(1-x)\right]_0^1}_{=0} + \frac{1}{2n^2\pi^2}\int_0^1 \cos(n\pi x)(1-2x)\,dx$$

$$= \frac{1}{2n^3\pi^3}\underbrace{\left[\sin(n\pi x)(1-2x)\right]_0^1}_{=0} + \frac{1}{n^3\pi^3}\int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{1}{n^4\pi^4}\left[-\cos(n\pi x)\right]_0^1 = \frac{1}{n^4\pi^4}\left(1-(-1)^n\right) = \begin{cases} 2/n^4\pi^4, & n=2k+1 \\ 0, & \text{ellers} \end{cases}$$

Ser at $B_n \neq 0$ kun når $A_n \neq 0$. Ved å introdusere $\mu_{kj} = 4k+2j+1$

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{j=0}^{1}\left(\frac{4(-1)^j}{\mu_{kj}^2\pi^2}\cos\left(\frac{\mu_{kj}\pi}{L}ct\right) + \frac{2}{\mu_{kj}^4\pi^4}\sin\left(\frac{\mu_{kj}\pi}{L}ct\right)\right)\sin\left(\frac{\mu_{kj}\pi}{L}x\right)$$

Oppgave 3

Teorem 4.3, med $c=2$

$$u(x,t) = \frac{1}{2}\left(\tanh(x+2t)^2 + \tanh(x-2t)^2\right) + \frac{1}{4}\int_{x-2t}^{x+2t}\sin(2x)\,dx$$

$$\frac{1}{4}\int_{x-2t}^{x+2t}\sin(2x)\,dx = \frac{1}{8}\Big[\cos(2x)\Big]_{x-2t}^{x+2t}$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2}\tanh(x+2t)^2 + \frac{1}{2}\tanh(x-2t)^2) + \frac{1}{8}\cos(2x+4t) - \frac{1}{8}\cos(2x-4t)$$