Matematikk 4N - Øving 10

Oppgave 1

a) $\frac{\partial^3 u}{\partial t^3} = x^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial^3 u}{\partial t^3} - x^2 \frac{\partial u}{\partial x^2} = 0$

• $(u+v)$:

$$\frac{\partial^3 (u+v)}{\partial t^3} - x^2 \frac{\partial^2 (u+v)}{\partial x^2} = \underbrace{\frac{\partial^3 u}{\partial t^3} - x^2 \frac{\partial^2 u}{\partial x^2}}_{\substack{=0 \\ \text{(gitt i oppgaven)}}} + \underbrace{\frac{\partial^3 v}{\partial t^3} - x^2 \frac{\partial^2 v}{\partial x^2}}_{=0}$$

$= 0 \Rightarrow$ ok!

• $c \cdot u$: $\frac{\partial^3 (c \cdot u)}{\partial t^3} - x^2 \frac{\partial^2 (c \cdot u)}{\partial x^2}$

$$= c\underbrace{\left(\frac{\partial^3 u}{\partial t^3} - x^2 \frac{\partial^2 u}{\partial x^2}\right)}_{=0} = 0 \Rightarrow \text{ok!}$$

Superposisjonsprinsippet holder for $\frac{\partial^3 u}{\partial t^3} = x^2 \frac{\partial^2 u}{\partial x^2}$

b) $\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial t} - u \frac{\partial u}{\partial x} = 0$

• $(u+v)$:

$$\frac{\partial (u+v)}{\partial t} - (u+v) \frac{\partial (u+v)}{\partial x}$$

$$= \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} - u \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} - v \frac{\partial v}{\partial x}$$

$$= \underbrace{\frac{\partial u}{\partial t} - u \frac{\partial u}{\partial x}}_{=0} + \underbrace{\frac{\partial v}{\partial t} - v \frac{\partial v}{\partial x}}_{=0} - u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x}$$

$$= u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \neq 0$$

Superposisjonsprinsippet holder ikke for $\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}$

c) $\frac{\partial^2 u}{\partial t \partial x} - t \frac{\partial u}{\partial t} = 0$

• $(u+v)$:

$$\frac{\partial^2 (u+v)}{\partial t \partial x} - t \frac{\partial (u+v)}{\partial t} = \underbrace{\frac{\partial^2 u}{\partial t \partial x} - t \frac{\partial u}{\partial t}}_{=0} + \underbrace{\frac{\partial^2 v}{\partial t \partial x} - t \frac{\partial v}{\partial t}}_{=0} = 0 \Rightarrow \text{ok!}$$

$(u+v)(0,t) = u(0,t) + v(0,t) = 0 + 0 = 0 \Rightarrow$ ok!

$\frac{\partial (u+v)}{\partial x}(1,t) = \frac{\partial u}{\partial x}(1,t) + \frac{\partial v}{\partial x}(1,t) = 0 + 0 = 0 \Rightarrow$ ok!

• $C \cdot u$:

$$\frac{\partial^2 (C \cdot u)}{\partial t \partial x} - t \frac{\partial (C \cdot u)}{\partial t} = C\Big(\underbrace{\frac{\partial^2 u}{\partial t \partial x} - t \frac{\partial u}{\partial t}}_{=0}\Big) = 0 \Rightarrow \text{ok!}$$

$C \cdot u(0,t) = C \cdot 0 = 0 \Rightarrow$ ok!

$C \cdot \frac{\partial u}{\partial x}(1,t) = C \cdot 0 = 0 \Rightarrow$ ok!

PDE tilfredsstiller superposisjonsprinsippet

d) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$

• $(u+v)$:

$$\frac{\partial (u+v)}{\partial t} - \frac{\partial^2 (u+v)}{\partial x^2} = \underbrace{\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}}_{=0} + \underbrace{\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}}_{=0} = 0 \Rightarrow \text{ok!}$$

– Ser at $u(0,t) = 0$ oppfylles

$(u+v)(1,t) = u(1,t) + v(1,t) = 5t + 5t = 10t \neq 5t \Rightarrow$ ikke ok!

$\Rightarrow$ PDE tilfredsstiller ikke superposisjonsprinsippet

Oppgave 2

a) Teorem 5,1 i Nome sine notater:

Varmelikningen: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

med randkrav: $u(0,t) = u(L,t) = 0$

og initialkrav: $u(x,0) = f(x)$

løses av $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$

hvor $A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$

Her $u(x,0) = f(x) = 4\sin\left(\frac{5\pi x}{L}\right) + 7\sin\left(\frac{11\pi x}{L}\right)$

$$\Rightarrow A_n = \frac{2}{L}\left(\underbrace{\int_0^L 4\sin\left(\frac{5\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx}_{\substack{=0 \text{ når } n\neq 5 \\ = 5\frac{L}{2} \text{ når } n=5}} + \underbrace{\int_0^L 7\sin\left(\frac{11\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx}_{\substack{=0 \text{ når } n\neq 11 \\ = 7\frac{L}{2} \text{ når } n=11}}\right)$$

$\Rightarrow A_5 = 4,\ A_{11} = 7,\ A_n = 0\ \forall\, n \neq 5 \cup n \neq 11$

$$\Rightarrow \underline{\underline{u(x,t) = 4\sin\left(\frac{5\pi x}{L}\right)e^{-\left(\frac{5\pi}{L}\right)^2 t} + 7\sin\left(\frac{11\pi x}{L}\right)e^{-\left(\frac{11\pi}{L}\right)^2 t}}}$$

$\uparrow$
$c=1$, Teorem 5,1

b) La $u(x,t)$ være løsningen på $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1$

Fra hint, sjekker $v(x,t) = u(x,t) + x(x-L)/2$

$$\frac{\partial v}{\partial t} - \frac{\partial v^2}{\partial x^2} = \frac{\partial u}{\partial t} - \frac{\partial(x(x-L)/2)}{\partial t} - \frac{\partial u^2}{\partial x^2} - \frac{\partial^2(x(x-L)/2)}{\partial x^2}$$

$$= \underbrace{\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}}_{\substack{=1 \\ \text{(fra likningen)}}} - \underbrace{\frac{\partial(x(x-L)/2)}{\partial t}}_{=0} - \frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2} - \frac{xL}{2}\right) = 1 - 0 - 1 = 0$$

$\Rightarrow \frac{\partial v}{\partial t} - \frac{\partial v^2}{\partial x^2}$ er homogen.

- $\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0$ er homogen

- $v(0,t) = u(0,t) + 0(0-L)/2$
$= 0 + 0$
$= 0$

- $v(L,t) = u(L,t) + L\underbrace{(L-L)}_{=0}/2$
$= 0 + 0$
$= 0$

- $v(x,0) = u(x,0) + x(x-L)/2$
$\Rightarrow f(x) = 4\sin\left(\frac{5\pi x}{L}\right) + 7\sin\left(\frac{11\pi x}{L}\right) + x(x-L)/2$

Kan bruke Teorem 5.1 igjen, men må finne ny A_n:

$$A_n = \frac{2}{L}\Bigg(\underbrace{\int_0^L 4\sin\left(\frac{5\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx + \int_0^L 7\sin\left(\frac{11\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}_{\text{Tilsvarende a)}}$$

$$+ \underbrace{\int_0^L \frac{x(x-L)}{2}\sin\left(\frac{n\pi x}{L}\right)dx}_{:=I}\Bigg)$$

$\frac{d}{dx}\left(\frac{x^2}{2} - \frac{xL}{2}\right)$
$= x - \frac{L}{2}$

$$I = \int_0^L \frac{x(x-L)}{2}\sin\left(\frac{n\pi x}{L}\right)dx$$

Delvis integrasjon $w = \frac{x(x-L)}{2}$, $z' = \sin\left(\frac{n\pi x}{L}\right)$

$$= \left[-\frac{L}{n\pi}\frac{x(x-L)}{2}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{L}{n\pi}\int_0^L\left(x-\frac{L}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dx,$$

$w = x - \frac{L}{2}$, $z' = \cos\left(\frac{n\pi x}{L}\right)$

$$= 0 + \frac{L}{n\pi}\left(\underbrace{\left[\frac{L}{n\pi}\left(x-\frac{L}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\right]_0^L}_{=0} - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= -\frac{L^2}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = -\frac{L^2}{n^2\pi^2}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{L^3}{n^3\pi^3}\left(\cos(n\pi) - \cos(0)\right) = \frac{L^3}{n^3\pi^3}\left((-1)^n - 1\right)$$

Dermed er

$$A_5 = 4 + \frac{2}{L} \cdot \frac{L^3}{5^3\pi^3}\left((-1)^5 - 1\right)$$
$$= 4 + \frac{2L^2}{125\pi^3} \cdot (-2)$$
$$= 4 - \frac{4L^2}{125\pi^3}$$

$$A_{11} = 7 + \frac{2}{L} \cdot \frac{L^3}{11^3\pi^3}\left((-1)^{11} - 1\right)$$
$$= 7 + \frac{4L^2}{1331\pi^3}$$

$$A_n = \frac{2L^2}{n^3\pi^3}\left[(-1)^n - 1\right] \quad n \neq 7, 11$$

Dermed er $v(x,t)$:

$$v(x,t) = \left[4 - \frac{4L^2}{125\pi^3}\right] \sin\left(\frac{5\pi x}{L}\right) e^{-\left(\frac{5\pi}{L}\right)^2 t} + \left[7 - \frac{4L^2}{1331\pi^3}\right] \sin\left(\frac{11\pi x}{L}\right) e^{-\left(\frac{11\pi}{L}\right)^2 t}$$
$$+ \sum_{n=1,\, n\neq 7,11}^{\infty} \frac{2L^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

Siden $v(x,t) = u(x,t) + \frac{x(x-L)}{2}$, er:

$$\underline{\underline{u(x,t) = \left[4 - \frac{4L^2}{125\pi^3}\right] \sin\left(\frac{5\pi x}{L}\right) e^{-\left(\frac{5\pi}{L}\right)^2 t} + \left[7 - \frac{4L^2}{1331\pi^3}\right] \sin\left(\frac{11\pi x}{L}\right) e^{-\left(\frac{11\pi}{L}\right)^2 t}}}$$
$$\underline{\underline{+ \sum_{n=1,\, n\neq 7,11}^{\infty} \frac{2L^2}{n^3\pi^3}\left[(-1)^n - 1\right] - \frac{x(x-L)}{2}}}$$

Oppgave 3

a) Trigonometrisk identitet:

$$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right)$$

$$\Rightarrow \int_0^{\pi} \cos\left(\left(n+\tfrac{1}{2}\right)x\right)\cos\left(\left(m+\tfrac{1}{2}\right)x\right)dx$$

$$= \frac{1}{2}\int_0^{\pi} \cos((n-m)x)\,dx + \frac{1}{2}\int_0^{\pi} \cos((n+m+1))\,dx$$

$n=m$:

$$= \frac{1}{2}\int_0^{\pi} \underbrace{\cos(0)}_{=1}dx + \frac{1}{2}\int_0^{\pi} \cos((2n+1)x)\,dx$$

$$= \frac{1}{2}\left[x\right]_0^{\pi} + \frac{1}{2}\left[\frac{1}{2n+1}\sin(\underbrace{(2n+1)}_{\text{heltall}}x)\right]_0^{\pi}$$

$$= \frac{1}{2}\pi - 0 + \frac{1}{2(2n+1)}(0-0)$$

$$= \frac{1}{2}\pi$$

$n \neq m$: Får null da $\int_0^{\pi} \cos(px)\,dx = 0 \quad \forall\, p \in \mathbb{Z}$

$$\Rightarrow \int_0^{\pi} \cos\left(\left(n+\tfrac{1}{2}\right)x\right)\cos\left(\left(m+\tfrac{1}{2}\right)x\right)dx = \begin{cases} \frac{\pi}{2}, & n=m \\ 0, & n\neq m \end{cases}$$

b) Separasjon av variabler:

La $u(x,t) = F(x)\cdot G(t)$

Da: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \Rightarrow F(x)G'(t) = F''(x)G(t)$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)} = k$$

$$\Rightarrow \begin{cases} F'' = kF \\ G' = kG \end{cases}$$

- $G' = kG$

$\Rightarrow G(t) = Ae^{kt}$

- $F'' = kF$

1) $k=0 \Rightarrow F''(x)=0 \Rightarrow F(x) = Bx + C$

BC. $F'(0) = F(\pi) = 0$

$\Rightarrow B = 0 \cap B\pi + C = B = 0, \Rightarrow B = C = 0$, Trivell løsning

2) $k > 0$:

$F(x) = Be^{\sqrt{k}x} + Ce^{-\sqrt{k}x}$

$F'(x) = B\sqrt{k}e^{\sqrt{k}x} - C\sqrt{k}e^{-\sqrt{k}x}$

BC: • $F'(0) = 0 \Rightarrow B\sqrt{k} - C\sqrt{k} = \sqrt{k}(B-C) = 0$

$\Rightarrow B = C$

• $F(\pi) = 0 \Rightarrow B\underbrace{\left(e^{\sqrt{k}\pi} + e^{-\sqrt{k}\pi}\right)}_{>0 \ \forall k} = 0$

$\Rightarrow B = 0$

$\Rightarrow$ Trivell løsning

3) $k < 0$:

La $k = -\lambda^2$, da er $\lambda > 0$

$\Rightarrow F''(x) = -\lambda^2 F(x)$

$\Rightarrow F(x) = B\cos(\lambda x) + C\sin(\lambda x)$

$F'(x) = -B\lambda\sin(\lambda x) + C\lambda\cos(\lambda x)$

BC: • $F'(0) = \lambda(-B \cdot \sin(0) + C \cdot \cos 0) = \lambda C = 0 \Rightarrow C = 0$

• $F(\pi) = 0$

$\Rightarrow B\cos(\lambda\pi) = 0$

$B = 0$ gir trivell løsning, bruker $\cos(\lambda\pi) = 0$

$\Rightarrow \lambda_n \pi = \pi n + \frac{\pi}{2}$, $n \geq 0$

$\Rightarrow \lambda_n = n + \frac{1}{2}$

$\Rightarrow k = -(n+\frac{1}{2})^2$

Kombinerer løsningene:

$u_n(x,t) = A_n \cdot B_n \cos((n+\frac{1}{2})x) e^{-(n+\frac{1}{2})^2 t}$

La $A_n \cdot B_n = \tilde{A}_n$

$\Rightarrow$ Generell løsning:

$$u(x,t) = \sum_{n=0}^{\infty} \tilde{A}_n \cos((n+\tfrac{1}{2})x) e^{-(n+\frac{1}{2})^2 t}$$

• Initial condition:

$$u(x,0) = \sum_{n=0}^{\infty} \tilde{A}_n \cos((n+\tfrac{1}{2})x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Fant i a) at $\cos((n+\frac{1}{2})x)$ er ett ortonormalt system

$\Rightarrow \tilde{A}_n$ er fourierkoeffisientene med $f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$

$\Rightarrow \tilde{A}_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos((n+\frac{1}{2})x) dx$

$$\frac{\pi \tilde{A}_n}{2} = \int_0^{\pi/2} x \cos((n+\tfrac{1}{2})x)\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos((n+\tfrac{1}{2})x)\,dx$$

$u = f(x)$

$v' = \cos((n+\frac{1}{2})x)$

$$= \cancel{\left[\frac{1}{n+1/2} x \sin((n+1/2)x)\right]_0^{\pi/2}} - \frac{1}{n+1/2} \int_0^{\pi/2} \sin((n+1/2)x)\,dx$$

$$+ \cancel{\left[\frac{1}{n+1/2} (\pi - x) \sin((n+\tfrac{1}{2})x)\right]_{\pi/2}^{\pi}} + \frac{1}{n+1/2} \int_{\pi/2}^{\pi} \sin((n+\tfrac{1}{2})x)\,dx$$

$$= \left[\frac{1}{(n+1/2)^2}\cos\left(\left(n+\tfrac{1}{2}\right)x\right)\right]_0^{\pi/2} - \left[\frac{1}{(n+1/2)^2}\cos\left(\left(n+1/2\right)x\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{(n+1/2)^2}\left(\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi}{2}\right) - \underbrace{\cos(0)}_{=1} - \underbrace{\cos\Big(\underbrace{\left(n+\tfrac{1}{2}\right)\pi}_{=\frac{m\pi}{2},\ m \text{ er oddetall}}\Big)}_{=0} + \cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{(n+1/2)^2}\left(2\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi}{2}\right) - 1\right)$$

$$\Rightarrow \tilde{A}_n = \frac{2}{\pi(n+1/2)^2}\left(2\cos\left(\left(n+\tfrac{1}{2}\right)\frac{\pi}{2}\right) - 1\right)$$

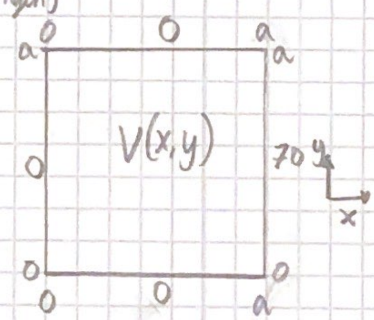
$\Rightarrow$ Løsningen blir:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{2}{\pi(n+1/2)^2}\left(2\cos\left(\left(n+1/2\right)\frac{\pi}{2}\right) - 1\right)\cos\left(\left(n+\tfrac{1}{2}\right)x\right)e^{-\left(n+\frac{1}{2}\right)^2 t}$$

Oppgave 4

(Antar at u løser likningen)

Hint: Definerer $v(x,y) = u(x,y) - 20$



$$\Rightarrow \begin{aligned} v(x,0) &= 0 & 0<x<a \\ v(x,a) &= 0 & 0<x<a \\ v(0,y) &= 0 & 0<y<a \\ v(a,y) &= 70 & 0<y<a \end{aligned}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} - \underbrace{\frac{\partial^2 20}{\partial x^2}}_{=0} + \frac{\partial^2 u}{\partial y^2} - \underbrace{\frac{\partial^2 20}{\partial y^2}}_{=0} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

BRUNNEN

Antar $v(x,t) = F(x)G(y)$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = F''(x)G(y) + F(x)G''(y) = 0$$

$$\Rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = k$$

$$\Rightarrow \begin{cases} F''(x) = kF(x) \\ G''(y) = -kG(x) \end{cases}$$

• $k=0$:

$$G''(y) = 0 \Rightarrow G(y) = Ay + B$$

BC. • $G(0) = 0$

$$A \cdot 0 + B = 0 \Rightarrow B = 0$$

• $G(a) = 0$

$$A \cdot a = 0 \Rightarrow A = 0$$

$\Rightarrow G(y) = 0$ Trivell løsning

• $k<0$:

$$G''(y) = -kG(y) = \lambda^2 G(y) \quad , \; k = -\lambda^2, \lambda > 0$$

$$\Rightarrow G(y) = Ae^{\lambda y} + Be^{-\lambda y}$$

BC • $G(0) = 0$

$$A + B = 0 \Rightarrow A = -B$$

• $G(a) = 0$

$A\underbrace{\left(e^{\lambda a} - e^{-\lambda a}\right)}_{>0 \; \forall a} = 0 \Rightarrow A = 0$ Trivell løsning

$k>0$:

$$G''(y) = -k\,G(y)$$

$$\Rightarrow \quad G(y) = A\cos(\sqrt{k}\,y) + B\sin(\sqrt{k}\,y)$$

BC. $\cdot\, G(0) = 0$

$$A\cdot\cos(0) + B\cdot\sin(0) = A = 0$$

$\cdot\, G(a) = 0$

$B\cdot\sin(\sqrt{k}\,a) = 0$, $B=0 \Rightarrow$ Trivtell løsning , bruker $\sin(\sqrt{k}a)=0$

$$\sin(\sqrt{k}\,a) = 0$$

$$\sqrt{k}\,a = n\pi$$

$$\underline{k = \left(\frac{n\pi}{a}\right)^2} \quad \Rightarrow G(y) = B_n \sin\left(\frac{n\pi}{a}y\right)$$

$$F''(x) = kF(x)$$

$$\Rightarrow \quad F(x) = Ce^{\sqrt{k}x} + De^{-\sqrt{k}x}$$

B.C: $F(0) = 0 \Rightarrow C = -D$

$$\Rightarrow F(x) = C\underbrace{\left(e^{\sqrt{k}x} - e^{-\sqrt{k}x}\right)}_{=2\cdot\sinh(\sqrt{k}x)}$$

$$\Rightarrow F(x) = 2C\sinh(\sqrt{k}x) = 2C\sinh\left(\frac{n\pi x}{a}\right)$$

Kombinerer: $V_n(x,y) = \tilde{A}_n \sinh\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)$, $\tilde{A}_n = B_n\cdot 2C$

$$\Rightarrow V(x,y) = \sum_{n=1}^{\infty} \tilde{A}_n \sinh\left(\frac{n\pi}{a}x\right)\sin\left(\frac{n\pi}{a}y\right)$$

B.C. $u(a,y) = 70$

$$\Rightarrow \sum_{n=1}^{\infty} \tilde{A_n} \sinh(n\pi) \sin\left(\frac{n\pi}{a} y\right) = 70$$

$\Rightarrow \tilde{A_n} \cdot \sinh(n\pi)$ er fourierkoeffisienten

$$\Rightarrow \tilde{A_n} \cdot \sinh(n\pi) = \frac{2}{a}\int_0^a 70 \sin\left(\frac{n\pi}{a} y\right) dy$$

$$\Rightarrow \tilde{A_n} = \frac{2 \cdot 70}{\cancel{a} \cdot \sinh(n\pi)} \left[-\frac{\cancel{a}}{n\pi} \cos\left(\frac{n\pi}{a} y\right)\right]_0^a$$

$$\tilde{A_n} = \frac{140}{n\pi \sinh(n\pi)} \left(-\cos(n\pi) + \cos(0)\right)$$

$$\underline{\tilde{A_n} = \frac{140}{n\pi \sinh(n\pi)} \left(1 - (-1)^n\right)}$$

$\Rightarrow$ Ettersom $v(x,y) = u(x,y) - 20$, vil:

$$\underline{\underline{u(x,y) = 20 + \sum_{n=1}^{\infty} \frac{140\left(1-(-1)^n\right)}{n\pi \cdot \sinh(n\pi)} \cdot \sinh\left(\frac{n\pi}{a} x\right) \cdot \sin\left(\frac{n\pi}{a} y\right)}}$$

Ekstra: <u>20, hvis alle kantene holder 20 grader, vil også steady state temperaturen være 20 grader</u>

Oppgave 5

a) Gjort digitalt, se bilde

b) $$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-y^2} dy$$, substituerer $u = -y$

$\Rightarrow du = -dy$

$y = -x \Rightarrow u = x$

$y = 0 \Rightarrow u = 0$

$$= \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-(-u)^2} \cdot (-du)$$

$$= -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} du$$

$$= -\mathrm{erf}(x) \quad \Rightarrow \underline{\mathrm{erf}(-x) = -\mathrm{erf}(x) \text{ QED.}}$$

c) Teorem 5.2 i Nomes notater

Varmelikningen $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

med initialkrav $u(x,0) = f(x)$

løses av $u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) e^{\frac{-(x-v)^2}{4c^2 t}} dv$

Med $c=1$, får vi:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v)\, e^{\frac{-(x-v)^2}{4t}} dv$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-1}^{1} e^{\frac{-(x-v)^2}{4t}} dv \text{ , la } y = \frac{x-v}{\sqrt{4t}}$$

$$\boxed{\begin{array}{l} \Rightarrow \frac{dy}{dv} = -\frac{1}{\sqrt{4t}} \\ \Rightarrow dv = -2\sqrt{t}\, dy \\ v=1 \Rightarrow y = \frac{x-1}{2\sqrt{t}} \\ v=-1 \Rightarrow y = \frac{x+1}{2\sqrt{t}} \end{array}}$$

$$= -\frac{\cancel{2}\sqrt{t}}{\cancel{2}\sqrt{\pi}\sqrt{t}} \int_{\frac{x+1}{2\sqrt{t}}}^{\frac{x-1}{2\sqrt{t}}} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-1}{2\sqrt{t}}}^{\frac{x+1}{2\sqrt{t}}} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+1}{2\sqrt{t}}} e^{-y^2} dy - \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-1}{2\sqrt{t}}} e^{-y^2} dy$$

$$= \underline{\underline{\frac{1}{2}\left(\operatorname{erf}\left(\frac{x+1}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-1}{2\sqrt{t}}\right)\right)}}$$

BRUNNEN

d) Fra google, integralformen er gitt som:

$$u(x,t) = \int_0^\infty [A(p)\cos(px) + B(p)\sin(px)]\, e^{-c^2 \cdot p^2 t}\, dp$$

$$\boxed{\begin{array}{l} u(x,0) = \frac{\sin x}{x} \\ f(x) = \frac{\sin x}{x} \end{array}}$$

→ Finner fourierkoeffisientene $A(p)$ og $B(p)$ $\left(e^{-p^2 \cdot 0} = 1\right)$

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(px)\, dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin(x)}{x}\cos(px)}_{\frac{odd}{odd}\,\cdot\, even \,=\, even}\, dx$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin(x)}{x}\cos(px)\,dx$$

Hint: Fouriertransformasjonen (cosinus) til $g(p)$ er $H(v) = \frac{2}{\pi}\frac{\sin(v)}{v}$

$$\Rightarrow F_c(g(p)) = H(v) = \frac{2}{\pi}\frac{\sin(v)}{v}$$

Den inverse transformen er da:

$$F_c^{-1}(H(v)) = g(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{2}{\pi}\frac{\sin(v)}{v}\cos(pv)\,dv$$

$$\Rightarrow g(p) = A(p)\cdot\sqrt{\frac{2}{\pi}} \Rightarrow \underline{A(p) = \sqrt{\frac{\pi}{2}}\cdot g(p)}$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(px)\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\sin x}{x}\cdot\sin(px)}_{\frac{odd}{odd}\cdot odd = odd}\,dx = 0,\ B(p) = 0$$

$$\Rightarrow u(x,t) = \int_0^\infty \sqrt{\frac{\pi}{2}}\, g(p)\cos(px)\, e^{-p^2 t}\, dp$$

$$\underline{\underline{u(x,t) = \sqrt{\frac{\pi}{2}}\int_0^1 \cos(px)\, e^{-p^2 t}\, dp}}$$