## Øving 1 Matematikk 4N

### Oppgave 1

a) $\vec{x} \in \mathbb{R}^n$

Vil vise $\lim_{p\to\infty} \|\vec{x}\|_p = \max_{1\le i\le n} |x_i|$

*Antakelse: Finnes unik $j$ slik at for alle $i \neq j$, så $|x_i| < |x_j|$

$\hookrightarrow \max_{1\le i\le n} |x_i| = |x_j| = M$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = M\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)^{1/p} = e^{\ln\left(M\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)^{1/p}\right)}$$

$$= e^{\frac{1}{p}\cdot\ln\left(M\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)\right)} = e^{\frac{1}{p}}\cdot M\cdot\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p$$

$$\lim_{p\to\infty}\|\vec{x}\|_p = M\cdot\lim_{p\to\infty}\left(e^{\frac{1}{p}}\cdot\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)$$

- $\lim_{p\to\infty} e^{\frac{1}{p}} = e^0 = 1$
- Ettersom $\frac{|x_i|}{M} < 1$ for $i \neq j$, og $\frac{|x_i|}{M} = 1$ kun én gang $(i=j)$

  Vil $\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p \to 1$ når $p\to\infty$

$$\Rightarrow \lim_{p\to\infty}\left(e^{\frac{1}{p}}\cdot\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right) = e^0\cdot 1 = 1$$

Dermed: $\lim_{p\to\infty}\|\vec{x}\|_p = \lim_{p\to\infty} M\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)^{1/p} = M\cdot 1 = M$

Ettersom $M = \max_{1\le i\le n}|x_i|$

$$\Rightarrow \underline{\underline{\lim_{p\to\infty}\|\vec{x}\| = \max_{1\le i\le n}|x_i|}} \quad \blacksquare$$

(Her har jeg glemt å inkludere dersom $\vec{x} = \vec{0}$, men da vil $\|\vec{x}\|_p = \max_{1\le i\le n}|x_i|$ for alle $p$. se b))

b) Dersom $\vec{x}=\vec{0}$: $\|\vec{x}\|_p = \|\vec{0}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = 0$

$$\max_{1\leq i\leq n} |x_i| = 0$$

$$\Rightarrow \underline{\underline{\|\vec{x}\|_p = \max_{1\leq i\leq n} |x_i|}}$$

Dersom $\vec{x}\neq\vec{0}$ finnes det en $M\neq 0$, hvor $M=\max_{1\leq i\leq n}|x_i|$

Fra a) har vi:

$$\|\vec{x}\|_p = M\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)^{1/p}$$

- Minste mulige verdi for $\|\vec{x}\|_p$ er dersom alle $x_i$ uten om én ($x_j=M$) er lik 0. Da er

$$\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)^{1/p} = \left(\left(\frac{M}{M}\right)^p\right)^{1/p} = 1$$

$$\Rightarrow \underline{\|\vec{x}\|_p = M\cdot 1 = M}$$

- Største mulige verdi er dersom alle $x_i$ er like, altså er alle $x_i=M$. Da er

$$\left(\sum_{i=1}^{n}\left(\frac{|x_i|}{M}\right)^p\right)^{1/p} = \left(\sum_{i=1}^{n}\left(\frac{M}{M}\right)^p\right)^{1/p} = \left(\sum_{i=1}^{n} 1\right)^{1/p} = n^{1/p}$$

$$\Rightarrow \underline{\|\vec{x}\|_p = M n^{1/p}}$$

Dermed: $M \leq \|\vec{x}\|_p \leq M\cdot n^{1/p}$, ved å ta $\lim_{p\to\infty}$

$$M \leq \lim_{p\to\infty}\|\vec{x}\|_p \leq M$$

Ved skvisteoremet er da

$$\underline{\underline{\lim_{p\to\infty}\|\vec{x}\|_p = M = \max_{1\leq i\leq n}|x_i|}} \quad \blacksquare$$

c) Gjort på PC. Lagt ved i egen fil.

## Oppgave 2

$p_1 = 1: \quad \underline{w_1 = 1} \Rightarrow q_1 = \frac{1}{\|1\|} = 1, \quad \underline{q_1 = 1}$

$p_2 = x: \quad w_1 = x - \langle x, 1\rangle \cdot 1$

$= x - \int_0^1 x \cdot 1\,dx$

$\underline{= x - \frac{1}{2}} \Rightarrow q_2 = \frac{x-\frac{1}{2}}{\|x-\frac{1}{2}\|} = \frac{x-\frac{1}{2}}{\sqrt{\int_0^1 (x-\frac{1}{2})^2 dx}} = \frac{x-\frac{1}{2}}{1/\sqrt{12}}$

$q_2 = \sqrt{12}(x-\frac{1}{2})$

$\underline{q_2 = \sqrt{3}(2x-1)}$

$p_3 = x^2: \quad w_3 = x^2 - \langle x^2, \sqrt{3}(2x-1)\rangle \sqrt{3}(2x-1) - \langle x^2, 1\rangle \cdot 1$

$= x^2 - 3(2x-1)\int_0^1 (2x^3 - x^2)\,dx - \int_0^1 x^2\,dx$

$= x^2 - 3(2x-1)\cdot\frac{1}{6} - \frac{1}{3}$

$= x^2 - x + \frac{1}{2} - \frac{1}{3}$

$\underline{= x^2 - x + \frac{1}{6}} \Rightarrow q_3 = \frac{x^2-x+\frac{1}{6}}{\|x^2-x+\frac{1}{6}\|}$

$= \frac{x^2-x+\frac{1}{6}}{\sqrt{\int_0^1 (x^2-x+\frac{1}{6})^2 dx}}$

$= \frac{x^2-x+\frac{1}{6}}{1/\sqrt{180}}$

$\underline{= \sqrt{5}(6x^2-6x+1)}$

$\underline{\underline{\{q_1, q_2, q_3\} = \{1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1)\}}}$

Oppgave 3

Et par definisjoner:

(1): $p = m-n,\ q = m+n \quad,\ m,n \in \mathbb{Z} \Rightarrow p,q \in \mathbb{Z}$

(2): $\sin(2k\pi) = 0$ for alle $k \in \mathbb{Z}$

(3): $\cos(2k\pi) = 1$ for alle $k \in \mathbb{Z}$

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a) $g_0(x) = \cos(0 \cdot x) = \cos 0 = 1$

$\langle g_0, g_0 \rangle = \int_0^{2\pi} 1\, dx = 2\pi$

$\underline{\underline{\langle g_0, g_0 \rangle = 2\pi}}$

b) $f_n(x) = \sin(nx)$

$$\langle f_n, f_n \rangle = \int_0^{2\pi} \sin(nx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi}\left(\cos(nx-nx) - \cos(nx+nx)\right)dx$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\cos(0) - \cos(2nx)\right)dx = \frac{1}{2}\int_0^{2\pi}\left(1 - \cos(2nx)\right)dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_0^{2\pi} = \frac{1}{2}\left((2\pi - 0) - \frac{1}{2n}(\sin(4n\pi) - \sin(0))\right)$$

$$\overset{(2)}{=} \frac{1}{2}\left(2\pi - \frac{1}{2n}(0-0)\right) = \underline{\pi}$$

$\underline{\underline{\langle f_n, f_n \rangle = \pi}}$

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c) $\langle f_m, f_n \rangle = \int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} (\cos(mx-nx) - \cos(mx+nx))\,dx$

$$\overset{(1)}{=} \frac{1}{2}\int_0^{2\pi} (\cos(px) - \cos(qx))\,dx$$

$$= \frac{1}{2}\left[\frac{1}{p}\sin(px) - \frac{1}{q}\sin(qx)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left(\frac{1}{p}(\sin(2p\pi) - \sin(0)) - \frac{1}{q}(\sin(2q\pi) - \sin(0))\right)$$

$$\overset{(2)}{=} \frac{1}{2}\left(\frac{1}{p}(0) - \frac{1}{q}(0)\right)$$

$$= \underline{0}$$

$\underline{\underline{\langle f_m, f_n \rangle = 0}}$

d)

$\langle f_m, g_n \rangle = \int_0^{2\pi} \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi} (\sin(mx+nx) + \sin(mx-nx))\,dx$

$$\overset{(1)}{=} \frac{1}{2}\int_0^{2\pi} (\sin(qx) + \sin(px))\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{q}\cos(qx) - \frac{1}{p}\cos(px)\right]_0^{2\pi}$$

$$= -\frac{1}{2}\left(\frac{1}{q}(\cos(2q\pi) - \cos(0)) + \frac{1}{p}(\cos(2p\pi - \cos(0))\right)$$

$$\overset{(3)}{=} -\frac{1}{2}\left(\frac{1}{q}(1-1) + \frac{1}{p}(1-1)\right)$$

$$= 0$$

$\underline{\underline{\langle f_m, g_n \rangle = 0}}$

## Oppgave 4

Kravene som må oppfylles for at $\|\cdot\|$ er en norm:

(1): $\|f\| \geq 0$ for alle $f \in C[0,1]$

(2): $\|f\| = 0$ kun dersom $f = 0$

(3): $\|\alpha \cdot f\| = |\alpha| \|f\|$ for alle $f \in C[0,1]$ og $\alpha \in \mathbb{R}$

(4): $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$ for alle $f_1, f_2 \in C[0,1]$

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a) $\|f\|_C = \max_{x \in [0,1]} |f(x)|$ Finner maksimal $|f(x)|$ for $x \in [0,1]$

(1): Ettersom alle $f \in C[0,1]$ er kontinuerlige og definerte for $x \in [0,1]$, følger det at (1) blir oppfylt

(2): La $f = 0 \Rightarrow \|0\|_C = \max_{x \in [0,1]} |f(x)| = 0$

Dersom $f \neq 0 \Rightarrow f(x) \neq 0$ for minst én $x \in [0,1]$

Dermed er $\|f\|_C = \max_{x \in [0,1]} |f(x)| \neq 0$ dersom $f \neq 0 \Rightarrow$ (2) ok!

(3): $\|\alpha \cdot f\|_C = \max_{x \in [0,1]} |\alpha \cdot f(x)| = |\alpha| \cdot \max_{x \in [0,1]} |f(x)| = |\alpha| \|f\|_C$

$\Rightarrow \|\alpha \cdot f\|_C = |\alpha| \|f\|_C \Rightarrow$ (3) ok!

(4): $\|f_1 + f_2\|_C = \max_{x \in [0,1]} |f_1(x) + f_2(x)| \leq \underbrace{\max_{x \in [0,1]} f_1(x) + \max_{x \in [0,1]} f_2(x)}_{= \|f_1\|_C + \|f_2\|_C}$

$\Rightarrow \|f_1 + f_2\|_C \leq \|f_1\|_C + \|f_2\|_C \Rightarrow$ (4) ok!

$\|\cdot\|_C$ oppfyller alle kravene og er derfor en norm

b) Det er lett å se (og vise) at $\|\cdot\|_*$ oppfyller normkravene (1), (3) og (4)

Derimot oppfyller den ikke (2)

Alle $f$ er definert for $x \in [0,1]$, men ettersom $\|f\|_* = \max_{x\in[0,0.5]} |f(x)|$ ser på max absoluttverdi for $f(x)$ når $x \in [0,0.5]$, kan det være andre funksjoner enn $f(x)=0$ som gir $\|f\|_* = 0$

Eksempel: $\tilde{f}(x) = \begin{cases} 0, & x \in [0,0.5] \\ x-\frac{1}{2}, & x \in \langle 0.5, 1] \end{cases}$

$\|\tilde{f}\|_* = 0 \land \tilde{f}(x) \neq 0 \Rightarrow \|\cdot\|_*$ oppfyller ikke (2)

$\|\cdot\|_*$ er ikke en norm