

Innlevering 5 - Matematikk 3

Kapittel 10

Oppgave 1: Løs $\det(A - \lambda I_n) = 0$ for egenverdier λ
egenvektor: $(A - \lambda I_n) \cdot \vec{x} = 0$ løs for $\vec{x}$

a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, A - \lambda I_n = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$$\det(A - \lambda I_n) = (1-\lambda)^2 - 4 = 0$$

$$(1-\lambda)^2 = 4$$

$$1-\lambda = \pm 2$$

$$\lambda_1 = -1 \vee \lambda_2 = 3$$

$\mathbb{R}$

1)

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(A + I_n) \cdot \vec{x}_1 = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \left(\text{Spenn av denne uten } \vec{0} \right)$$



2) $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} (A - 3I_n) \vec{x}_2 = 0 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbb{R}$ — r —

$$\lambda_1 = -1, \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot r$$

$$\lambda_2 = 3, \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot r$$

$\mathbb{R}$

r står for vilkårlig tall

$$b) \det(A - \lambda I_n) = \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)(1-\lambda)(-\lambda) - 2(2 \cdot (-\lambda)) = 0$$

$$(1-2\lambda+\lambda^2)(-\lambda) + 4\lambda = 0$$

$$-\lambda(1-2\lambda+\lambda^2-4) = 0$$

$$\underline{\lambda_1 = 0} \quad \vee \quad \lambda^2 - 2\lambda - 3 = 0$$

$$\underline{\lambda_2 = 3} \quad \vee \quad \underline{\lambda_3 = -1}$$

R

$$0: \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0} \Rightarrow \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \vec{0}$$

$$\Rightarrow x=0, y=0, z=t$$

$$\Rightarrow \underline{\vec{X}_1 = \text{sp} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

R

$$2) \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A\vec{x} = \begin{bmatrix} x-y \\ z \\ 0 \end{bmatrix} = \vec{0}$$

$$\Rightarrow x=y \text{ og } z=0$$

$$\Rightarrow \underline{\vec{X}_2 = \text{sp} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

R

$$3) \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A\vec{x} = \begin{bmatrix} x+y \\ z \\ 0 \end{bmatrix} = \vec{0}$$

$$x+y=0 \Rightarrow x=-y \text{ or } z=0$$

$$\Rightarrow \vec{x}_3 = \text{sp} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} \quad \mathbb{R}$$

$$\lambda_1 = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot r$$

$$\lambda_2 = 3 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot r$$

$$\lambda_3 = -1 \Rightarrow \vec{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot r$$

$$c) \det(A - \lambda I_n) = \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 = 0$$

$$\Rightarrow \underline{\lambda = 0}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A\vec{x} = \begin{bmatrix} y \\ 0 \end{bmatrix}^T = \vec{0}$$

$$y=0 \Rightarrow \underline{\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot r} \quad \mathbb{R}$$

$$\underline{\lambda = 0 \Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot r}$$

$$d) \det(A - \lambda I_3) = \det \begin{pmatrix} 4-\lambda & 2 & 3 \\ -1 & 1-\lambda & -3 \\ 2 & 4 & 9-\lambda \end{pmatrix} = 0$$

Lös:

$$\begin{aligned} &= (4-\lambda)((1-\lambda)(9-\lambda)+12) - 2((\lambda-9)+6) + 3((-4)-2(1-\lambda)) \\ &= (4-\lambda)(\lambda^2-10\lambda+21) - 2\lambda+6-12-6+6\lambda \\ &= 4\lambda^2-40\lambda+84-\lambda^3+10\lambda^2-21\lambda-2\lambda+6-12-6+6\lambda \\ &= \underline{-\lambda^3+14\lambda^2-57\lambda+72} \end{aligned}$$

$$-(\lambda^3 + 14\lambda^2 + 57\lambda - 72) = 0$$

Testet litt i verktøy
fant faktor $(\lambda - 3)$

$$-(\lambda - 3)(\lambda^2 + 11\lambda + 24) = 0$$

$$-(\lambda - 3)^2(\lambda - 8) = 0$$

$$\underline{\lambda_1 = \lambda_2 = 3, \lambda_3 = 8}$$

$$(\lambda^2 + 14\lambda + 57\lambda - 72)(\lambda - 3) = \underline{\lambda^2 - 11\lambda + 24}$$

$$-(\lambda^2 - 3\lambda^2)$$

$$-11\lambda^2 + 57\lambda$$

$$-(11\lambda^2 + 33\lambda)$$

$$24\lambda - 72$$

$$-(24\lambda - 72)$$

1) og 2)

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A\vec{x}_1 = \begin{bmatrix} x+2y+3z \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$\Rightarrow x + 2y + 3z = 0$$

$$\Rightarrow x = -2y - 3z$$

innfører parametre $t=y$ og $s=z$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot t + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot s \Rightarrow *$$

$$3) \begin{bmatrix} -4 & 2 & 3 \\ -1 & -7 & -3 \\ 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 \\ -4 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 \\ 0 & 30 & 15 \\ 0 & -10 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 7 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow A\vec{x}_3 = \begin{bmatrix} x - \frac{1}{2}z \\ y + \frac{1}{2}z \\ 0 \end{bmatrix} = \vec{0}$$

$$\left. \begin{array}{l} x = \frac{1}{2}z \\ y = -\frac{1}{2}z \end{array} \right\} \Rightarrow x = -y = \frac{1}{2}z$$

$$\Rightarrow \vec{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot r$$

$$\underline{\lambda_1 = \lambda_2 = 3 \Rightarrow \vec{x}_1 = \vec{x}_2 = \text{sp} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$$\underline{\lambda_3 = 8 \Rightarrow \vec{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot r}$$

Oppgave 2.

a) $\det(A - \lambda I_2) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0$
 $\lambda^2 = -1 \Rightarrow \lambda = \pm\sqrt{-1} = \pm i$
ingen reell løsning R

b)

For en matrise er eventuell rotasjon gitt slik at:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

i vårt tilfelle: $\cos \alpha = 0$
 $\sin \alpha = -1$ } $\Rightarrow \alpha = \frac{3\pi}{2}$ (mot klokke)
 $= \frac{\pi}{2}$ (med klokke)

rotasjon $\frac{\pi}{2} \Rightarrow 90^\circ$ på vektor $\Rightarrow$ umulig å finne reell C s.a. $A\vec{x} = C \cdot \vec{x}$
 $\Rightarrow$ Finnes ingen egenverdi. R (umulig å løse f.eks $C \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ da vektorene er ortogonale og ikke parallelle)

Oppgave 3:

a) $\vec{e}_1$ rotert $\theta = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ (enhetssirkel) R
 $\vec{e}_2$ rotert $\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ R
(se for seg enhetssirkel, roter akser 90° med klokke)
 $\Rightarrow x = \cos \theta, 90^\circ$ rotasjon $\Rightarrow x = -\sin \theta$
 $y = \sin \theta, 90^\circ$ rotasjon $\Rightarrow y = \cos \theta$

b) $T_\theta = [\vec{e}_1 \text{ rotert } \theta; \vec{e}_2 \text{ rotert } \theta] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
 $T_\theta(\vec{e}_1)$

$$T_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} R$$

c) Når vi enten roterer vektoren med en halv eller en hel runde, har vi egenverdi. $\lambda = \pm 1$, da er vektor etter matrisetransformasjon lik en skalar (± 1) $\cdot$ vektor før.

$$\Rightarrow \underline{\theta = n \cdot \pi} \text{ der } n \in \mathbb{Z} \quad \checkmark$$

Oppgave 4:

$$a) \det(A - \lambda I_4) = \det \begin{pmatrix} 1-\lambda & 2 & 3 & 4 \\ 0 & 2-\lambda & 3 & 4 \\ 0 & 0 & -5-\lambda & 0 \\ 0 & 0 & 0 & 77-\lambda \end{pmatrix}$$

Øvre triangulær matrise $\Rightarrow \det(A - \lambda I_4) = \underline{(1-\lambda)(2-\lambda)(-5-\lambda)(77-\lambda)}$

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$$\Rightarrow \underline{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -5, \lambda_4 = 77} \quad \checkmark$$

b) $\lambda_1 = 1$

$$(A - I_4) \cdot \vec{x}_1 = \vec{0} \Rightarrow \begin{bmatrix} 0 & 2 & 3 & 4 & | & 0 \\ 0 & 1 & 3 & 4 & | & 0 \\ 0 & 0 & -6 & 0 & | & 0 \\ 0 & 0 & 0 & 76 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix} \Rightarrow x_1: \text{fri}, x_2 = x_3 = x_4 = 0$$

$$\Rightarrow \text{egenrom: } \text{sp} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \checkmark$$

$$\lambda_2 = 2 \quad (A - 2I_4) \cdot \vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -1 & 2 & 3 & 4 & | & 0 \\ 0 & 0 & 3 & 4 & | & 0 \\ 0 & 0 & -7 & 0 & | & 0 \\ 0 & 0 & 0 & 75 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix} \Rightarrow x_1 = -2x_2, x_3 = x_4 = 0 \Rightarrow \text{egenrom: } \text{sp} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \checkmark$$

$$\lambda_3 = -5$$

$$(A + 5I_4)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} 6 & 2 & 3 & 4 & 1 & 0 \\ 0 & 7 & 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 82 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/14 & 0 & 1 & 0 \\ 0 & 1 & 3/7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow x_4 = 0, x_3 \text{ er variabel, } x_1 = -\frac{5}{14} x_3$$

$$x_2 = -\frac{3}{7} x_3$$

$$x_3 = 1 \Rightarrow x_1 = -\frac{5}{14}, x_2 = -\frac{3}{7}$$

Skaleret
 $x_3 = 14 \Rightarrow x_1 = -5, x_2 = -6 \Rightarrow \text{Egenrom: } \text{Sp} \left\{ \begin{bmatrix} -5 \\ -6 \\ 14 \\ 0 \end{bmatrix} \right\} \mathbb{R}$

$$\lambda_4 = 77$$

$$(A - 77I_4)\vec{x} = \vec{0} \Rightarrow \begin{bmatrix} -76 & 2 & 3 & 4 & 1 & 0 \\ 0 & -75 & 3 & 4 & 1 & 0 \\ 0 & 0 & -82 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -76 & 77 & 0 & 0 & 1 & 0 \\ 0 & 75 & 0 & -4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

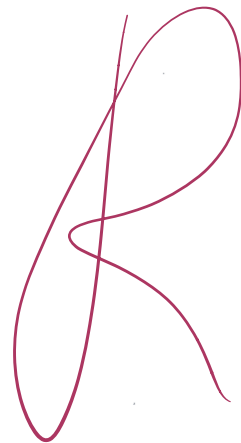
$$\Rightarrow \begin{cases} 76x_1 = 77x_2 \\ 4x_4 = 75x_2 \\ x_3 = 0 \end{cases} \Rightarrow \text{eigenrom: } \text{Sp} \left\{ \begin{bmatrix} 77 \\ 76 \\ 0 \\ 1425 \end{bmatrix} \right\} \mathbb{R}$$

Svar: $\lambda_1 = 1 \Rightarrow \text{eigenrom } \text{Sp} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$

$\lambda_2 = 2 \Rightarrow \text{eigenrom } \text{Sp} \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$

$\lambda_3 = -5 \Rightarrow \text{eigenrom } \text{Sp} \left(\begin{bmatrix} -5 \\ -6 \\ 14 \\ 0 \end{bmatrix} \right)$

$\lambda_4 = 77 \Rightarrow \text{eigenrom } \text{Sp} \left(\begin{bmatrix} 77 \\ 76 \\ 0 \\ 1425 \end{bmatrix} \right)$



c) Nei, det er det ikke. Vi hadde en triangulær matrise, som er et spesialtilfelle. Det gjorde ting lettere.

Oppgave 5:

a) $\vec{0}$ er ikke egenvektor

$$A \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \\ 80 \end{bmatrix} = 20 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \lambda = 20 \text{ egenvektor: } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad R$$

$$A \cdot \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 102 \\ 74 \\ 66 \\ 18 \end{bmatrix}, 74 \neq 66 \Rightarrow \text{ingen egenvektor}$$

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 28 \\ 6 \\ 4 \\ 2 \end{bmatrix}, \text{ ikke egenvektor}$$

$$A \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \\ 40 \\ 60 \end{bmatrix} = 20 \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \lambda = 20 \text{ egenvektor: } \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \quad R$$

$$A \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 120 \\ 80 \\ 40 \\ 40 \end{bmatrix} = 40 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \lambda = 40 \text{ egenvektor: } \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix} \quad R$$

$$A \cdot \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 160 \\ 120 \\ 80 \\ 40 \end{bmatrix} = 40 \cdot \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \lambda = 40 \text{ egenvektor: } \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad R$$

Egenvektorer i settet: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ og $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ R

b) For hver λ -verdi, er vektorene fra a) lin. uavhengige

$$\Rightarrow \lambda_1 = 20 \Rightarrow \text{Egenrom: } \text{Sp} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$$

$$\lambda_2 = 40 \Rightarrow \text{Egenrom: } \text{Sp} \left(\begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right) \quad R$$

Oppgave 6: La c være en vilkårlig egenverdi for A

Derom egenverdi: $A\vec{x} = c\vec{x} \Rightarrow \vec{x} \neq \vec{0}$

$$\text{Siden } A^2 = A \Rightarrow A^2\vec{x} = A\vec{x} \quad / \text{ ved egenverdi:} \\ c^2\vec{x} = c\vec{x}$$

$$c^2\vec{x} - c\vec{x} = 0 \\ c(c-1)\vec{x} = 0$$

$$\underline{c=0 \vee c=1}$$

Får altså at en slik matrise får egenverdier 0 og 1

Oppgave 7: Derom A er inverterbar, $A\vec{x} = \vec{0}$ har kun løsningen $\vec{x} = \vec{0}$

Derom A har egenverdien 0, finnes en vektor $\vec{v} \neq \vec{0}$ s.a.

$$A \cdot \vec{v} = 0 \cdot \vec{v} = \vec{0} \Rightarrow A\vec{x} = \vec{0} \text{ har løsningen } \vec{v} = \vec{0}$$

Dermed er matrisen ikke inverterbar per def. 

Kapittel 11

Oppgave 1: a) $\begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}$

$$\det(A - \lambda I_3) = \det \begin{bmatrix} 3-\lambda & -1 & 2 \\ 3 & -1-\lambda & 6 \\ -2 & 2 & -2-\lambda \end{bmatrix}$$

$$\begin{aligned} &= (3-\lambda)(-1-\lambda)(-2-\lambda) + (3(-2-\lambda) + 12) + 2(6 + 2(-1-\lambda)) \\ &= (3-\lambda)(\lambda^2 + 3\lambda + 10) + 6 - 3\lambda + 12 + 12 + 4 - 4\lambda \\ &= 3\lambda^2 + 9\lambda + 30 - \lambda^3 - 3\lambda^2 + 10\lambda - 7\lambda + 14 \\ &= -\lambda^3 + 12\lambda - 16 \end{aligned}$$

$$\det(A - \lambda I_3) = -\lambda^3 + 12\lambda - 16 = 0$$

$$(\lambda - 2)(-\lambda^2 - 2\lambda + 8) = 0$$

$$-(\lambda - 2)(\lambda^2 + 2\lambda - 8) = 0$$

$$-(\lambda - 2)^2(\lambda + 4) = 0$$

$$\lambda_1 = 2 \text{ og } \lambda_2 = -4$$

$$\underline{\lambda_1 = 2}$$

Prøv og feil $\Rightarrow \lambda = 2$ løser

$$-\lambda^3 + 12\lambda - 16 : (\lambda - 2) = -\lambda^2 + 2\lambda + 8$$

$$-(-\lambda^2 + 2\lambda^2)$$

$$\leftarrow -2\lambda^2 + 12\lambda$$

$$-(-2\lambda^2 + 4\lambda)$$

$$18\lambda - 16$$

$$-(18\lambda - 16)$$

$$0$$

$$(A - 2I_3)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 3 & -3 & 6 & 0 \\ -2 & 2 & -4 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow x_1 - x_2 + 2x_3 = 0 \Rightarrow x_1 = x_2 - 2x_3, x_2 = t, x_3 = s$$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} s \Rightarrow \text{Egenrom } \lambda_1 = 2 : \text{sp} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

R

$$\lambda_2 = -4 \quad (A + 4I_3)\vec{x} = \vec{0}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 7 & -1 & 2 & 0 \\ 3 & 3 & 6 & 0 \\ -2 & 2 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left. \begin{array}{l} 2x_1 + x_3 = 0 \Rightarrow x_1 = -\frac{1}{2}x_3 \\ 2x_2 + 3x_3 = 0 \Rightarrow x_2 = -\frac{3}{2}x_3 \end{array} \right\} \Rightarrow \text{La } x_3 = t$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -1/2 \\ -3/2 \\ 1 \end{bmatrix} t \Rightarrow \text{Egenrom: } \text{sp} \left(\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right) \text{ for } \lambda_2 = -4$$

Vi har tre egenvektorer som er lineært uavhengige, med to egenverdier. R

$\Rightarrow$ Matrisen er diagonaliserbar R

b) Har begynt å gå tom for papir i boka. Herifra forenkles ting jeg har gjort for.

$$\det(A - \lambda I_4) = \det \left(\begin{bmatrix} -\lambda & -1 & 1 & 5 \\ 1 & -\lambda & 2 & 6 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 4 & -\lambda \end{bmatrix} \right) \left(\begin{array}{l} \text{ser at kun } \begin{vmatrix} 3-\lambda & 0 \\ 4 & -\lambda \end{vmatrix} \\ \text{gir verdi} \Rightarrow \text{kan se bort} \\ \text{ifra kombinasjoner som} \\ \text{ender i andre} \end{array} \right)$$

$$= -\lambda \begin{vmatrix} -\lambda & 2 & 6 \\ 0 & 3-\lambda & 0 \\ 0 & 4 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 & 5 \\ 0 & 3-\lambda & 0 \\ 0 & 4 & -\lambda \end{vmatrix} + 0 - 0$$

$$= -\lambda^2((3-\lambda)(\lambda)) + 1(3-\lambda)(-\lambda) = -\lambda^4 + 3\lambda^3 + \lambda^2 - 3\lambda$$

$$= (-\lambda)(\lambda^3 - 3\lambda^2 - \lambda - 3) = (-\lambda)(\lambda - 3)(\lambda^2 + 1)$$

$$\Rightarrow \underline{\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = -i, \lambda_4 = i}$$

R

$$\lambda_1 = 0$$

$$\begin{bmatrix} 0 & -1 & 1 & 5 & 10 \\ 1 & 0 & 2 & 6 & 10 \\ 0 & 0 & 3 & 0 & 10 \\ 0 & 0 & 4 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 6 & 10 \\ 0 & -1 & 0 & 5 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -6 \\ 5 \\ 0 \\ 1 \end{bmatrix} \cdot t$$

$$\Rightarrow \text{Egenrom: } \text{Sp} \left(\begin{bmatrix} -6 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right) \quad \mathbb{R}$$

$$\lambda_2 = 3$$

$$\begin{bmatrix} -3 & -1 & 1 & 5 & 10 \\ 1 & -3 & 2 & 6 & 10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -39/40 & 0 \\ 0 & 1 & 0 & -113/40 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} 39 \\ 113 \\ 30 \\ 40 \end{bmatrix} t$$

$$\Rightarrow \text{Egenrom: } \text{Sp} \left(\begin{bmatrix} 39 \\ 113 \\ 30 \\ 40 \end{bmatrix} \right) \quad \mathbb{R}$$

$$\lambda_3 = -i$$

$$\begin{bmatrix} i & -1 & 1 & 5 & 10 \\ 1 & i & 2 & 6 & 10 \\ 0 & 0 & 3+i & 0 & 10 \\ 0 & 0 & 4 & i & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & i & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot t$$

$$\Rightarrow \text{Egenrom: } \text{Sp} \left(\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \quad \mathbb{R}$$

$$\lambda_4 = i$$

$$\begin{bmatrix} -i & -1 & 1 & 5 & 10 \\ 1 & -i & 2 & 6 & 10 \\ 0 & 0 & 3-i & 0 & 10 \\ 0 & 0 & 4 & -i & 10 \end{bmatrix} \sim \begin{bmatrix} -i & -1 & 0 & 0 & 10 \\ 1 & -i & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -i & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot t \Rightarrow \text{Egenrom: } \text{Sp} \left(\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) \quad \mathbb{R}$$

Matrisen er 4×4 , vi har 4 egenverdier. Matrisen er diagonaliserbar ved teorem 11.3

$\mathbb{R}$

Oppgave 2: For å finne D må vi ha egenverdiene til A
 For å finne P må vi ha egenvektorene til A

$$\det(A - I_2 \cdot \lambda) = \det \begin{pmatrix} 1-\lambda & 1-i \\ 1+i & -1-\lambda \end{pmatrix} = \lambda^2 - 1 - (1 - (-1)) = 0$$

$$\lambda^2 - 3 = 0$$

$$\lambda = \pm \sqrt{3}$$

$$\Rightarrow D = \begin{bmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

Egenvektorer: $\lambda = -\sqrt{3}$

$$(A + \sqrt{3}I_2) = \begin{bmatrix} 1+\sqrt{3} & 1-i \\ 1+i & \sqrt{3}-1 \end{bmatrix}$$

$$(A + \sqrt{3}I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 1+\sqrt{3} & 1-i & 0 \\ 1+i & \sqrt{3}-1 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1+\sqrt{3} & 1-i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow (1+\sqrt{3})x_1 + (1-i)x_2 = 0$$

$$x_1 = \frac{(i-1)}{(1+\sqrt{3})} x_2 \quad \text{La } t = x_2$$

$$\Rightarrow \text{Egenvektorer: } \vec{x} = \begin{bmatrix} i-1 \\ 1+\sqrt{3} \end{bmatrix} \cdot t \quad \text{der } t \text{ er en parameter } \in \mathbb{R}$$

Egenvektorer: $\lambda = \sqrt{3}$

$$(A - \sqrt{3}I_2)\vec{x} = \vec{0} \Rightarrow \left[\begin{array}{cc|c} 1-\sqrt{3} & 1-i & 0 \\ 1+i & -1-\sqrt{3} & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1-\sqrt{3} & 1-i & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow (1-\sqrt{3})x_1 + (1-i)x_2 = 0 \Rightarrow x_1 = \frac{(i-1)}{(1-\sqrt{3})} x_2 \quad \text{La } x_2 = t$$

$$\Rightarrow \text{Egenvektorer: } \vec{x} = \begin{bmatrix} i-1 \\ 1-\sqrt{3} \end{bmatrix} \cdot t$$

Vi får $D = \begin{bmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$ og $P = \begin{bmatrix} i-1 & i-1 \\ 1+\sqrt{3} & 1-\sqrt{3} \end{bmatrix}$ R

Oppgave 3:

løser $\det(A - \lambda I_2) = 0$

$$\begin{vmatrix} r_1 - \lambda & z \\ \bar{z} & r_2 - \lambda \end{vmatrix} = (r_1 - \lambda)(r_2 - \lambda) - z\bar{z} = 0$$

$$= r_1 r_2 - (r_1 + r_2)\lambda + \lambda^2 - z\bar{z} = 0$$

$$\lambda^2 - (r_1 + r_2)\lambda + (r_1 r_2 - z\bar{z}) = 0$$

$$\Rightarrow \lambda = \frac{r_1 + r_2 \pm \sqrt{(r_1 + r_2)^2 - 4 \cdot 1 \cdot (r_1 r_2 - z\bar{z})}}{2}$$

Vil alltid for tall i $\mathbb{R}$

fordi $(r_1 - r_2)^2 > 0$ og $z\bar{z} \in \mathbb{R}$

$\Rightarrow$ alltid positivt tall i $\mathbb{R}$ under rottegn

$\Rightarrow$ λ alltid i $\mathbb{R}$ R

$$\lambda = \frac{r_1 + r_2 \pm \sqrt{(r_1 - r_2)^2 + 4z\bar{z}}}{2}$$

Oppgave 4

$$\det(A - \lambda I_3) = \begin{vmatrix} a - \lambda & b & 0 \\ b & a - \lambda & 0 \\ 0 & 0 & a - b - \lambda \end{vmatrix}$$

$$= (a - b - \lambda)((a - \lambda)^2 - b^2)$$

$$a - b - \lambda = 0 \quad \vee \quad (a - \lambda)^2 - b^2 = 0$$

$$\underline{\lambda = a - b}$$

$$(a - \lambda)^2 = b^2$$

$$a - \lambda = \pm b$$

$$\underline{\lambda = a \pm b}$$

$$\Rightarrow \underline{\lambda = a \pm b}$$

$$(A - (a-b)I_3) = \begin{bmatrix} a-(a-b) & b & 0 \\ b & a-(a+b) & 0 \\ 0 & 0 & (a-b)-(a-b) \end{bmatrix} = \begin{bmatrix} b & b & 0 \\ b & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{X} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot s \Rightarrow \text{Egenrom: } \text{sp} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$(A - (a+b)I_3) = \begin{bmatrix} a-(a+b) & b & 0 \\ b & a-(a+b) & 0 \\ 0 & 0 & (a-b)-(a+b) \end{bmatrix} = \begin{bmatrix} -b & b & 0 \\ b & -b & 0 \\ 0 & 0 & -2b \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{X} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot t \Rightarrow \text{Egenrom: } \text{sp} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \quad \mathbb{R}$$

Eigenverdier: $\lambda = (a-b)$, $\lambda = (a+b)$

Tilhørende egenrom: $\text{sp} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ og $\text{sp} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$ $\mathbb{R}$

Den er diagonaliserbar, 3 lineært uavhengige egenvektorer. (3x3-matrise)

Oppgave 5:

$$a) \quad A = [T(1) \mid T(x) \mid T(x^2)] \quad \begin{aligned} T(1) &= 0+1=1 \\ T(x) &= (x+1) \cdot 1 + x = 2x+1 \\ T(x^2) &= (x+1)2x + x^2 = 3x^2+2x \end{aligned}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$\mathbb{R}$

b) Vi har en øvre triangular matrise

$$\Rightarrow \det(A - I_3 \lambda) = (1 - \lambda)(2 - \lambda)(3 - \lambda) = 0$$

$$\Rightarrow \underline{\lambda = \{1, 2, 3\}}$$

$$\underline{\lambda = 1} \Rightarrow (A - I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot t}$$

$$\underline{\lambda = 2} \Rightarrow (A - 2I_3) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot t}$$

$$\underline{\lambda = 3} \Rightarrow (A - 3I_3) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot t}$$

$(x_1 = x_3 = \frac{1}{2}x_2)$

Eigenverdier: $\lambda = \{1, 2, 3\}$, Egenvektorer: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

3x3-matrise med 3 ulike eigenverdier $\Rightarrow$ diagonaliserbar

Oppgave 6: a) For en 2x2-rotasjonsmatrise er

$$(Oppgave 10.3 \Rightarrow) R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \stackrel{\text{vært tilfeldig}}{=} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\cos \theta = \frac{\sqrt{3}}{2}, \sin \theta = \frac{1}{2}$$

$$\Downarrow \\ \underline{\underline{\theta = \frac{\pi}{6}}}$$

$$b) \det \begin{pmatrix} \frac{\sqrt{3}}{2} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{pmatrix} = \left(\frac{\sqrt{3}}{2} - \lambda\right)^2 - \frac{1}{4} = 0$$

$$\lambda^2 - \sqrt{3}\lambda + \frac{3}{4} - \frac{1}{4} = 0$$

$$\lambda^2 - \sqrt{3}\lambda + 1 = 0$$

$$\lambda = \frac{\sqrt{3} \pm \sqrt{(-\sqrt{3})^2 - 4 \cdot 1 \cdot 1}}{2} = \frac{\sqrt{3} \pm i}{2} \quad R$$

$$\lambda = \frac{\sqrt{3}}{2} + \frac{i}{2} \Rightarrow \begin{bmatrix} -\frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix} \cdot t}$$

$$\lambda = \frac{\sqrt{3}}{2} - \frac{i}{2} \Rightarrow \begin{bmatrix} \frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow \underline{\vec{x} = \begin{bmatrix} i \\ -1 \end{bmatrix} \cdot t}$$

$$\text{Egenverdier: } \lambda = \frac{\sqrt{3} \pm i}{2}, \text{ Egenvektorer: } \underline{\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ -1 \end{bmatrix}} \quad R$$

$$c) \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} i \\ -1 \end{bmatrix}, \text{ per def vil } T(\vec{v}_1) = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \vec{v}_1 = \lambda_1 \cdot \vec{v}_1$$

$$T(\vec{v}_2) = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) \vec{v}_2 = \lambda_2 \cdot \vec{v}_2$$

Hvilket gir ny matrise B:

$$B = \begin{bmatrix} T(\vec{v}_1) & T(\vec{v}_2) \\ \uparrow & \uparrow \\ (\vec{e}_1 \cdot \lambda) & (\vec{e}_2 \cdot \lambda) \end{bmatrix} = \underline{\underline{B = \begin{bmatrix} \frac{\sqrt{3}}{2} + \frac{i}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} - \frac{i}{2} \end{bmatrix}}} \quad R$$

Bra!!

A stylized handwritten signature or set of initials in red ink, featuring a large loop and a trailing flourish.