Innlevering 4 - Matematikk 2

Oppgave 1 $\vec{F}(x,y,z) = (xe^{2z}, ye^{2z}, -e^{2z}) = (F_1, F_2, F_3)$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = e^{2z} + e^{2z} - 2e^{2z} = 0 \Rightarrow \underline{\operatorname{div} \vec{F} = 0}$$

$$\underline{\underline{\operatorname{div} \vec{F} = 0 \Rightarrow \vec{F} \text{ er divergensfritt}}} \ \blacksquare$$

- Finner $\vec{G}$ og $\vec{H}$ utifra definisjonen av curl. Gjør valg slik at jeg får svarene fra fasit.

La $\vec{G}(x,y,z) = (G_1, G_2, G_3)$

curl $\vec{G} = \vec{F}$ dersom:

(I) $\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = F_1 = xe^{2z}$

(II) $\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = F_2 = ye^{2z}$

(III) $\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3 = -e^{2z}$

Velger $G_3 = 0$:

$\Rightarrow$ (I): $-\frac{\partial G_2}{\partial z} = xe^{2z}$

$\underline{G_2 = -\frac{1}{2}xe^{2z} + C_1(x,y)}$

$\Rightarrow$ (II): $\frac{\partial G_1}{\partial z} = ye^{2z}$

$\underline{G_1 = \frac{1}{2}ye^{2z} + C_2(x,y)}$

Setter inn G_1 og G_2 i (III):

$-\frac{1}{2}e^{2z} + \frac{\partial C_1}{\partial x} - \frac{1}{2}e^{2z} - \frac{\partial C_2}{\partial y} = -e^{2z}$

$\Rightarrow \frac{\partial C_1}{\partial x} - \frac{\partial C_2}{\partial y} = 0$, Velger $C_1 = C_2 = 0$

$\Rightarrow \underline{\underline{\vec{G} = (\frac{1}{2}ye^{2z}, -\frac{1}{2}xe^{2z}, 0)}}$

La $\vec{H}(x,y,z) = (H_1, H_2, H_3)$

curl $\vec{H} = \vec{F}$ dersom:

(IV): $\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} = F_1 = xe^{2z}$

(V): $\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} = F_2 = ye^{2z}$

(VI): $\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = F_3 = -e^{2z}$

Velger $H_2 = 0$:

$\Rightarrow$ (IV): $\frac{\partial H_3}{\partial y} = xe^{2z}$

$\underline{H_3 = xye^{2z} + C_3(x,z)}$

(VI): $-\frac{\partial H_1}{\partial y} = -e^{2z}$

$\underline{H_1 = ye^{2z} + C_4(x,z)}$

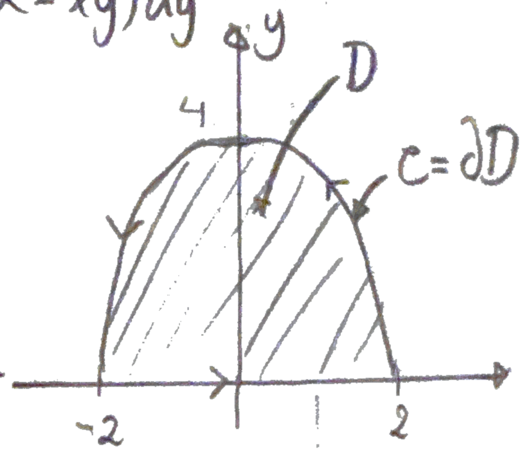
Setter H_1, H_3 inn i (V):

$2ye^{2z} + \frac{\partial C_4}{\partial z} - ye^{2z} - \frac{\partial C_3}{\partial x} = 1ye^{2z}$

$\Rightarrow \frac{\partial C_4}{\partial z} - \frac{\partial C_3}{\partial x} = 0$, velger $C_3 = C_4 = 0$

$\Rightarrow \underline{\underline{\vec{H} = (ye^{2z}, 0, xye^{2z})}}$

Oppgave 2: $\oint_C (\sin x + y^2)\,dx + (\cos x - xy)\,dy$



La $P = \sin x + y^2$

og $Q = \cos x - xy$

Da er $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\sin x - y - 2y = \underline{-\sin x - 3y}$

Ved Greens teorem:

$$\oint_C P\,dx + Q\,dy = \int_{-2}^{2}\int_0^{4-x^2} (-\sin x - 3y)\,dy\,dx =$$

$$= \underbrace{\int_{-2}^{2}\int_0^{4-x^2} -\sin x\,dy\,dx}_{=I_1} - \underbrace{\int_{-2}^{2}\int_0^{4-x^2} 3y\,dy\,dx}_{=I_2} = I_1 + I_2 = *$$

$$I_1 = \int_{-2}^{2}\int_0^{4-x^2} -\sin x\,dy\,dx = \int_{-2}^{2} -(4-x^2)(\sin x)\,dx$$

Ser vanskelig ut, men dersom $p(x)$ e odde, vil $\int_{-a}^{a} p(x)\,dx = 0$

La $f(x) = -(4-x^2)(\sin x)$, dersom $f(-x) = -f(x)$ er f odde.

$$f(-x) = -(4-(-x)^2)(\sin(-x)) = (4-x^2)(\sin x) = -f(x)$$

sin x er en odde funksjon

$\Rightarrow f(x)$ er odde

$\Rightarrow \underline{I_1 = 0}$

$$I_2 = \int_{-2}^{2}\int_0^{4-x^2} 3y\,dy\,dx = \int_{-2}^{2} \left[\frac{3}{2}y^2\right]_0^{4-x^2} dx = \frac{3}{2}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx$$

$$= \frac{3}{2}\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{-2}^{2} = \frac{3}{2}\left(\left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right)\right)$$

$$\Rightarrow I_2 = \frac{256}{5}$$

$$* = I_1 - I_2 = 0 - \frac{256}{5} = -\frac{256}{5}. \quad \underline{\underline{\oint_C P\,dx + Q\,dy = -\frac{256}{5}}}$$

Oppgave 3

a) $V_T = \iiint_T dV$

Sylinderkoordinater:

$x = r\cos\theta \qquad \theta \in [0, 2\pi]$

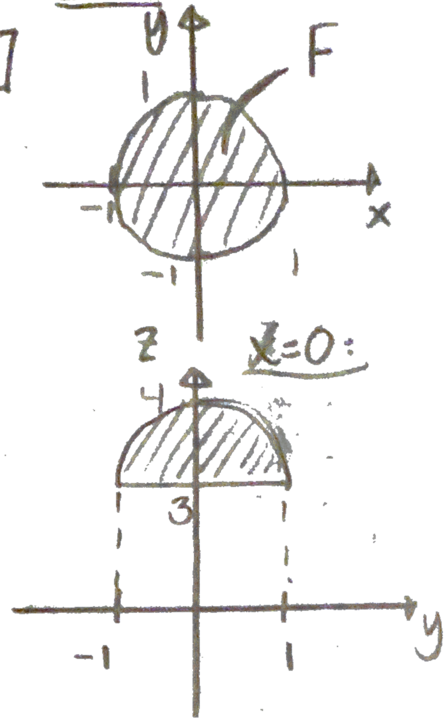
$y = r\sin\theta \qquad r \in [0, 1]$

fra uttrykk paraboloide:

$3 < z < 4 - r^2$

Skisse T:

Z=3:



Innfører sylinderkoordinater:

$$V_T = \int_0^{2\pi}\int_0^1\int_3^{4-r^2} r \; dz\,dr\,d\theta = 2\pi\int_0^1 (1-r^2)r\,dr$$

$$= 2\pi\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = 2\pi\cdot\frac{1}{4} = \underline{\frac{\pi}{2}}$$

$$\underline{\underline{V_T = \frac{\pi}{2}}}$$

b) S er ikke lukket, men ved å lukke den med flaten F (se skisse) får vi T, og kan bruke divergensteoremet:

$$\iint_T \vec{F}\cdot\vec{N}\,dS = \iiint_T \operatorname{div}\vec{F}\,dV = \iiint_T dV = \frac{\pi}{2}$$

$\boxed{\operatorname{div}\vec{F} = -2xy + 2xy + 1 = 1}$ fra a)

Siden S ikke har grunnflate F, vil:

$$\iint_S \vec{F}\cdot\hat{N}\,dS = \iint_T \vec{F}\cdot\hat{N}\,dS - \iint_F \vec{F}\cdot\hat{N}\,dS = \underline{\frac{\pi}{2} - \iint_F \vec{F}\cdot\hat{N}\,dS}$$

Bunnflaten F får normalvektor $\hat{N} = (0,0,-1)$, $\vec{F} = (-x^2y, xy^2, 3)$ $\boxed{\text{Planet } z=3}$

$$\iint_F \vec{F}\cdot\hat{N}\,dS = \iint_F -3\,dS = -3\iint_F dA = \underline{-3\pi} \qquad \boxed{\text{Sirkel } r=1}$$

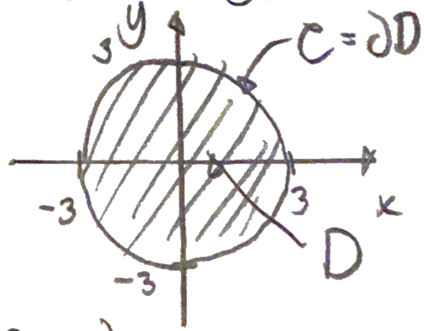
$$\Rightarrow \underline{\underline{\iint_S \vec{F}\cdot\hat{N}\,dS = \frac{\pi}{2} - (-3\pi) = \frac{7\pi}{2}}}$$

Oppgave 4: Finner skjæringskurven; setter z-uttrykkene like.

$$\sqrt{x^2+y^2+1} = \sqrt{10}$$

$$\mathcal{C}: \quad \underline{x^2+y^2 = 9}$$

Skisse $\mathcal{C}$:



Sirkelflaten D og kurven $\mathcal{C}$ oppfyller kravene til Stokes teorem:

$$\boxed{\oint_{\mathcal{C}} \vec{F}\, d\vec{r} = \iint_D (\text{curl}\, \vec{F}) \cdot \hat{N}\, dS}$$

Velger $\hat{N} = (0,0,1)$ ettersom denne står normalt på $z = \sqrt{10}$ og dermed også D

Siden $(\text{curl}\, \vec{F})$ prikkes med $\hat{N} = (0,0,1)$ blir kun 3. komponenten relevant.

La $\vec{F} = (x+y,\, 4x-y,\, z^2+xy) = (F_1, F_2, F_3)$

$$\Rightarrow (\text{curl}\, \vec{F}) \hat{N} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 4 - 1 = \underline{3}$$

$$\Rightarrow \oint_{\mathcal{C}} \vec{F}\, d\vec{r} = \iint_D (\text{curl}\, \vec{F}) \cdot \hat{N}\, dS = \iint_D 3\, dS = 3 \iint_D dA = 3 \cdot \text{Areal } D$$

$$= 3 \cdot \pi r^2 = 3 \cdot 3^2 \cdot \pi = \underline{27\pi}$$

$$\underline{\underline{\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 27\pi}}$$