Innlevering 3 - Matematikk 2

Oppgave 1: Vi kan splitte ulikheten til:

$$0 \leq \sqrt{x^2+y^2+z^2} < 1+z \quad , \quad z<0$$
$$0 \leq \sqrt{x^2+y^2+z^2} < 1-z \quad , \quad z\geq 0$$

Ser at legemet er symmetrisk om $z=0$, vi kan også se at massefunksjonen $\delta(x,y,z) = (x^2+y^2+z^2)^{-\frac{3}{4}}$ er symmetrisk om $z=0$
$\Rightarrow$ Vi kan regne ut massen av delen over $z=0$, og gange med 2.

Innfører sfæriske koordinater.

$\Rightarrow 0 < \rho < 1-\rho\cdot\cos\varphi, \; 0 \leq \varphi \leq \frac{\pi}{2}$ (her er $z>0$)

$\Rightarrow M: \; (\rho^2)^{-3/4} = \rho^{-3/2}$

$\Rightarrow \; 0 \leq \theta \leq 2\pi$ (z=0 gir sirkel i xy-planet.)

$$\text{Masse}_T = 2\cdot\int_0^{2\pi}\int_0^{\pi/2}\int_0^{1/1+\cos\varphi} \rho^{-3/2}\cdot\rho^2\cdot\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2\cdot 2\pi \int_0^{\pi/2}\int_0^{1/1+\cos\varphi} \sqrt{\rho}\cdot\sin\varphi \, d\rho\, d\varphi$$

$$= 4\pi \int_0^{\pi/2} \left[\frac{2}{3}\rho^{3/2}\right]_0^{\frac{1}{1+\cos\varphi}} \cdot \sin\varphi \, d\varphi$$

$$\boxed{\begin{array}{l} *: u = 1+\cos\varphi \\ du = -\sin\varphi \, d\varphi \\ \varphi = \pi/2 \Rightarrow u=1 \\ \varphi = 0 \Rightarrow u = 2 \end{array}}$$

$$= 4\pi \int_0^{\pi/2} \frac{2}{3}\left(\frac{1}{1+\cos\varphi}\right)^{3/2} \cdot \sin\varphi \, d\varphi \overset{*}{=} 4\pi \int_2^1 \frac{2}{3} u^{-3/2} \frac{\sin\varphi}{(-\sin\varphi)} du$$

$$= 4\pi \int_2^1 -\frac{2}{3} u^{-3/2} du = 4\pi \int_1^2 \frac{2}{3} u^{-3/2} du$$

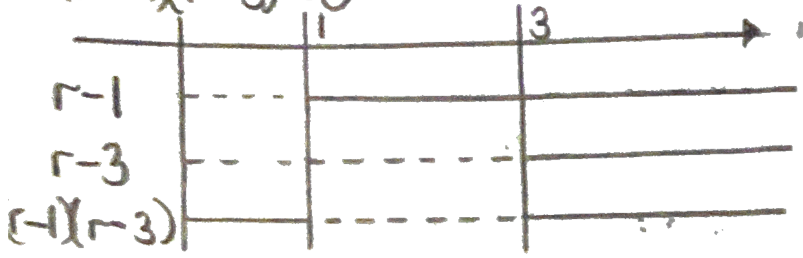
$$= 4\pi \cdot \left[\frac{2}{3}\cdot\frac{-2}{1}\cdot\frac{1}{\sqrt{u}}\right]_1^2 = \frac{8\pi}{3}\left(-\frac{2}{\sqrt{2}} - \left(-\frac{2}{\sqrt{1}}\right)\right) = \underline{\frac{8\pi}{3}(2-\sqrt{2})}$$

$$\underline{\text{Massen av } T = \frac{8\pi}{3}(2-\sqrt{2})}$$

Oppgave 2:

Sjekker hva r er ved $z=0$

$$(r-2)^2 \leq 1$$
$$r^2-4r+4 \leq 1$$
$$r^2-4r+3 \leq 0$$
$$(r-1)(r-3) \leq 0$$

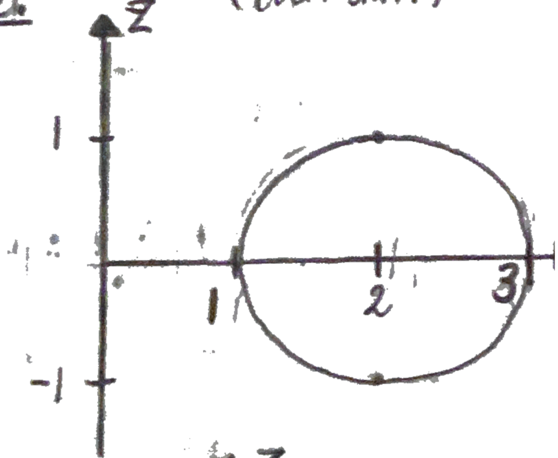


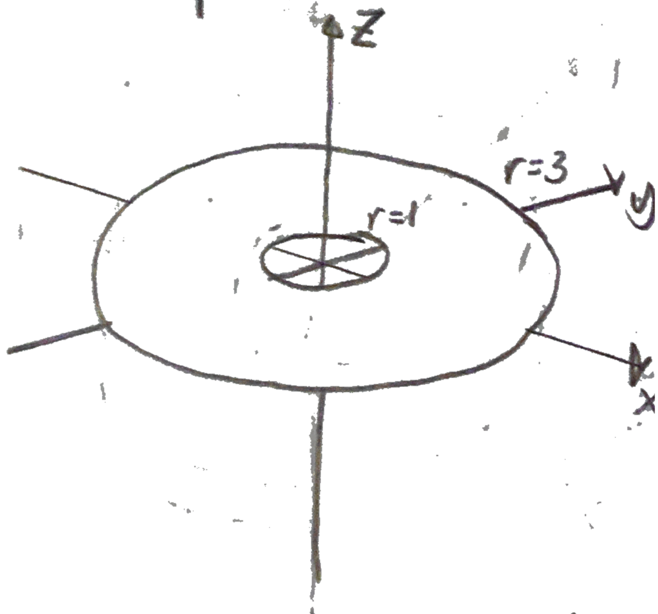
Oppfylt for $1 \leq r \leq 3$

Sjekker hva z er v/ $r-2=0$:

$$z^2 \leq 1$$
$$-1 \leq z \leq 1$$

Skisse: (tverrsnitt)





Ser at vi har en smultring

$$(r-2)^2+z^2 \leq 1 \qquad 0 \leq \theta \leq 2\pi$$

(ingen likninger begrenser θ)

$$z \leq \pm\sqrt{1-(r-2)^2}$$

Siden S er symmetrisk om $z=0$, kan vi la $z \leq \sqrt{1-(r-2)^2}$ og gange volumet med 2

$$V_S = 2 \cdot \int_0^{2\pi}\int_1^3\int_0^{\sqrt{1-(r-2)^2}} r \, dz\, dr\, d\theta$$

$$= 4\pi \int_1^3 r\cdot\sqrt{1-(r-2)^2}\, dr$$

Subst.

$$= 4\pi \int_{-\pi/2}^{\pi/2} (\sin u+2)\sqrt{1-(\sin u)^2}\cdot \cos u \, du$$

$$= 4\pi \left(\int_{-\pi/2}^{\pi/2} \cos^2 u \cdot \sin u \, du + \int_{-\pi/2}^{\pi/2} 2\cos^2 u \, du\right)$$

Subst: $r-2 = \sin u$

$$r = \sin u + 2$$
$$dr = \cos u \, du$$
$$r=3 \Rightarrow \sin u = 1 \Rightarrow u = \frac{\pi}{2}$$
$$r=1 \Rightarrow \sin u = -1 \Rightarrow u = -\frac{\pi}{2}$$

La
$$\left.\begin{aligned} I_1 &= \int_{-\pi/2}^{\pi/2} \cos^2 u \cdot \sin u \, du \\ I_2 &= \int_{-\pi/2}^{\pi/2} 2\cos^2 u \, du \end{aligned}\right\} \Rightarrow \underline{V_S = 4\pi(I_1+I_2)}$$

$$I_1 = \int_{-\pi/2}^{\pi/2} \cos^2 u \cdot \sin u \, du$$

$$\boxed{\begin{array}{l} *: \\ V = \cos u \\ dV = -\sin u \, du \\ u = \pi/2 \Rightarrow V = 0 \\ u = -\pi/2 \Rightarrow V = 0 \end{array}}$$

$$\overset{*}{=} \int_0^0 V^2 \cdot \frac{\sin u}{-\sin u} \, dV$$

$$= -\int_0^0 V^2 \, dV = \underline{0}$$

$$I_2 = \int_{-\pi/2}^{\pi/2} 2\cos^2 u \, du$$

$$\boxed{**: \cos^2 u = \frac{1+\cos(2u)}{2}}$$

$$\overset{**}{=} \int_{-\pi/2}^{\pi/2} 1 + \cos(2u) du = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) + \int_{-\pi/2}^{\pi/2} \cos(2u) \, du$$

$$\boxed{\begin{array}{l} ***: w = 2u \\ dw = 2 \, du \\ du = \frac{dw}{2} \\ u = \frac{\pi}{2} \Rightarrow w = \pi \\ u = -\frac{\pi}{2} \Rightarrow w = -\pi \end{array}}$$

$$\overset{***}{=} \pi + \frac{1}{2}\int_{-\pi}^{\pi} \cos w \, dw = \pi + \frac{1}{2}\left[\sin w\right]_{-\pi}^{\pi} = \underline{\pi}$$

$$\Rightarrow I_1 = 0, \; I_2 = \pi$$

$$V_S = 4\pi(I_1 + I_2) = 4\pi \cdot \pi = \underline{4\pi^2}$$

<u>Volumet av S är $4\pi^2$</u>

Oppgave 3: Konservativ => Det finnes $\varphi(x,y,z)$ s.a.

$$\nabla\varphi = \vec{F}_a(x,y,z)$$

$$\Rightarrow \nabla\varphi = (1+y^2,\ 2xy+z^2,\ 2a\cdot y\cdot z)$$

$$\frac{\partial\varphi}{\partial x} = 1+y^2 \Rightarrow \underline{\varphi(x,y,z) = x + xy^2 + C(y,z)}$$

$$\frac{\partial\varphi}{\partial y} = 2xy + \frac{\partial}{\partial y}C(y,z) = 2xy + z^2 + C(z)$$

$$\Rightarrow \underline{\varphi(x,y,z) = x + xy^2 + yz^2 + C(z)}$$

$$\frac{\partial\varphi}{\partial z} = 2yz + \frac{\partial}{\partial z}C(z) = 2ayz + C \quad \text{der } C\in\mathbb{R}$$

$$\Rightarrow \underline{\varphi(x,y,z) = x + xy^2 + yz^2 + C}$$

$\Rightarrow$ <u>For at dette skal gå, må $a=1$</u> $\Rightarrow \nabla\varphi = (1+y^2,\ 2xy+z^2,\ 2yz)$

$$\int_e \vec{F}_a\cdot d\vec{r} = \int_0^{1/2} \nabla\varphi(r(t))\cdot r'(t)\,dt = \int_0^{1/2}\frac{d}{dt}\nabla\varphi(r(t))\,dt$$

fund. teorem.

$$= \nabla\varphi(r(\tfrac{1}{2})) - \nabla\varphi(r(0)) = \nabla\varphi((\tfrac{1}{2},1,0)) - \nabla\varphi((0,0,1))$$

$$= (\tfrac{1}{2}+\tfrac{1}{2}+C) - (0+C) = 1 + C - C = \underline{1}$$

$$\Rightarrow \underline{\underline{\int_e \vec{F}_a\cdot d\vec{r} = 1 \quad ,\ a=1}}$$

Oppgave 4: Finner prosekjon ved å eliminere z, setter likning for paraboloiden inn i likning for kuleflaken.

$$x^2+y^2+\left(\tfrac{1}{2}(x^2+y^2)\right)^2=3$$

Gjør om til polerkoordinater (sylinder koordinater)

$$r^2+\tfrac{1}{4}r^4-3=0$$

$$\underbrace{(r^2+6)}_{\text{ingen reell løsning}}(r^2-2)=0 \Rightarrow r^2-2=0$$

$$r^2=2$$

Substituerer inn $x^2+y^2=r^2$

$$x^2+y^2=2$$

S er kurven innenfor dette

$$\Rightarrow \underline{x^2+y^2\leq 2}$$

$$z=\tfrac{1}{2}(x^2+y^2) \Rightarrow z-\tfrac{1}{2}x^2-\tfrac{1}{2}y^2=0$$

La $G(x,y,z)=z-\tfrac{1}{2}x^2-\tfrac{1}{2}y^2$

$$\Rightarrow \nabla G=(-x,-y,1)$$

Arealet er gitt ved:

$$A=\iint_S \frac{|\nabla G|}{\partial G/\partial z}\,dx\,dy=\iint_S \frac{\sqrt{(-x)^2+(-y)^2+1^2}}{1}\,dx\,dy$$

$$\overset{\text{Gjør om til polerkoordinater}}{=}\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{r^2+1}\cdot r\,dr\,d\theta$$

Subst.
$u=r^2+1$
$du=2r\,dr$
$r=\sqrt{2}\Rightarrow u=3$
$r=0\Rightarrow u=1$

$$\overset{\text{Subst.}}{=}2\pi\int_1^3\sqrt{u}\cdot\frac{\cancel{r}}{2\cancel{r}}\,du\,d\theta=\frac{2\pi}{2}\left[\frac{2}{3}u^{3/2}\right]_1^3=\underline{2\pi\left(\sqrt{3}-\tfrac{1}{3}\right)}$$

Arealet er $2\pi\left(\sqrt{3}-\tfrac{1}{3}\right)$