Matematikk 2 - Innlevering 2

Oppgave 1: a) $f(x,y) = 2x^2 - x^4 + y^2$

Ekstremalpunkt er gitt ved $\frac{\partial f}{\partial x} = 0$ og $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 4x - 4x^3$, $\frac{\partial f}{\partial x} = 0 \Rightarrow 4x - 4x^3 = 0$

$4x(1-x^2) = 0$

$\underline{x = 0 \quad \vee \quad x = \pm 1}$

$\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0$

$\underline{y = 0}$

Gir ekstremalpunktene: $(0,0)$, $(-1,0)$, $(1,0)$

Sjekker ved 2. derivertest :

$$\boxed{\begin{aligned} A &= \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2 \\ C &= \frac{\partial^2 f}{\partial y^2} = 2 \\ B &= \frac{\partial^2 f}{\partial x \partial y} = 0 \end{aligned}}$$

$D = AC - B^2$
$= 8 - 24x^2$

$\underline{(0,0)}$: $D = 8 > 0$, $A = 4 > 0 \Rightarrow$ Lokalt minimum

$\underline{(-1,0)}$: $D = -16 < 0 \Rightarrow$ Sadelpunkt

$\underline{(1,0)}$: $D = -16 < 0 \Rightarrow$ Sadelpunkt

Kritiske punkt: Sadelpunkt: $(-1,0)$ og $(1,0)$

Lokal minimumspunkt $(0,0)$

b) La $g(x,y) = x^4 + y^2 - 4$

Finner maks. og min. verdi der $\nabla f \parallel \nabla g$

Bruker Lagranges $\boxed{\nabla f = \lambda \nabla g}$ (Gir potensielle max/min pkt)

$\nabla f = \begin{pmatrix} 4x - 4x^3 \\ 2y \end{pmatrix}$

$\nabla g = \begin{pmatrix} 4x^3 \\ 2y \end{pmatrix}$ Får: I: $4x - 4x^3 = \lambda 4x^3$

II: $2y = \lambda 2y$

<u>$y = 0$:</u>

Klarer ikke løse likn. settet, men $g(x,0) = 0$ gir x-verdi:

$g(x,0) = x^4 - 4 = 0$

$\underline{x = \pm\sqrt{2}}$

Gir $(\pm\sqrt{2}, 0)$

$\underline{f(0, \pm\sqrt{2}) = 0}$

<u>$y \neq 0$:</u>

II $\Rightarrow \lambda = 1 \Rightarrow$

$\lambda = 1$ i I gir $4x - 4x^3 = 4x^3$

$4x - 8x^3 = 0$

$4x(1 - 2x^2) = 0$

$x = 0 \lor x = \pm\frac{1}{\sqrt{2}}$

$g(0,y) = 0$

$\underline{y = \pm 2}$

$g(\pm\frac{1}{\sqrt{2}}, y) = 0$

$\underline{y = \pm\frac{\sqrt{15}}{2}}$

$\underline{f(0, \pm 2) = 4}$

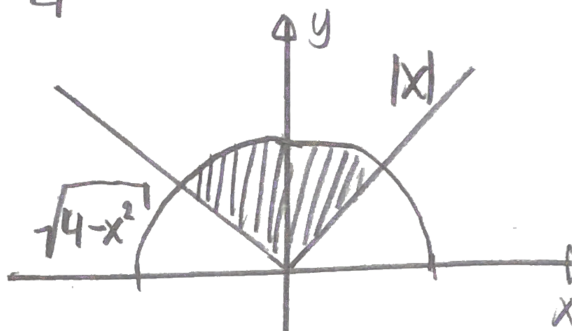
$\underline{f(\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{15}}{2}) = \frac{9}{2}, \quad f(-\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{15}}{2}) = \frac{9}{2}}$

<u>Største verdi for f på $x^4 + y^2 = 4$ er $9/2 = 4,5$</u>

<u>minste verdi er 0</u>

Oppgave 2:

Kan se at $A = \frac{A_{sirkel}}{4} = \pi$



Sjekker skjæring:

$$\sqrt{4-x^2} = |x|$$
$$4 - x^2 = x^2$$
$$2x^2 = 4$$
$$x = \pm\sqrt{2}$$

Ved å gjøre om til polare koordinater, kan vi integrere $\sqrt{4-x^2}$ fra skjæringspunkt til skjæringspunkt.

$$y = \sqrt{4-x^2}$$

$$y^2 = 4 - x^2$$
$$x^2 + y^2 = 4$$

La $x = r\cos\theta$, $y = r\sin\theta$ $\Big\}$ $r^2 = 4 \Rightarrow r = 2$

Grenser integral:

$x = r\cos\theta = 2\cos\theta = -\sqrt{2}$

$= \cos\theta = -\frac{\sqrt{2}}{2}$

$\theta = \frac{3\pi}{4}$

$x = 2\cos\theta = \sqrt{2}$

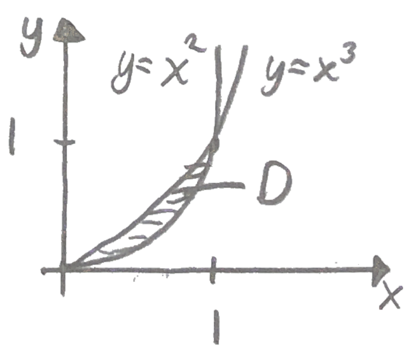
$\theta = \frac{\pi}{4}$

$$A = \int_{\pi/4}^{3\pi/4} 2\, d\theta = \left[2\theta\right]_{\pi/4}^{3\pi/4} = 2\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$$

$$\underline{\underline{A = \pi}}$$

Oppgave 3:

D: $y = x^2$, $y = x^3$



$$\iint_D (x - y^2)\, dA$$

Integrerer i y-retning

$$\int_0^1 \int_{x^3}^{x^2} (x - y^2)\, dy\, dx = \int_0^1 \left[xy - \frac{y^3}{3}\right]_{x^3}^{x^2} dx$$

$$= \int_0^1 \left(x^3 - x^4 - \frac{x^6}{3} + \frac{x^9}{3}\right) dx = \left[\frac{x^4}{4} - \frac{x^5}{5} - \frac{x^7}{21} + \frac{x^{10}}{30}\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{5} - \frac{1}{21} + \frac{1}{30} = \underline{\frac{1}{28}}$$

$$\underline{\iint_D (x - y^2)\, dA = \frac{1}{28}}$$

Oppgave 4

$x^2+y^2+z^2=4$
$x^2+(y-1)^2=1$

Gjør om til polarkoord.

La $x = r\cos\theta$
$y = r\sin\theta$

$$x^2+y^2+z^2=4 \overset{\text{Polar koord}}{\Rightarrow} r^2+z^2=4$$
$$\underline{z=\sqrt{4-r^2}}$$

$$x^2+(y-1)^2=1 \Rightarrow$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

$$x^2+y^2-2y+1=1 \quad \text{polar koord.}$$
$$x^2+y^2=2y \Rightarrow r^2 = 2r\sin\theta$$
$$\underline{r=2\sin\theta}$$

$$V = 4\int_0^{\pi/2}\int^{2\sin\theta}\sqrt{4-r^2}\,r\cdot dr\cdot d\theta$$

$$= 4\int_0^{\pi/2}\int_4^{4\cos^2\theta}\sqrt{u}\cdot \cancel{r}\,\frac{du}{-2\cancel{r}}\,d\theta$$

$$= -2\int_0^{\pi/2}\left[\frac{2u^{3/2}}{3}\right]_4^{4\cos^2\theta}$$

$$= -2\int_0^{\pi/2}\frac{2}{3}\left(8\cos^3\theta - 8\right) = \frac{32}{3}\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta$$

$$= \frac{32}{3}\left(\int_0^{\pi/2}d\theta - \int_0^1\frac{\cos^{\cancel{3}2}\theta\,dV}{\cancel{\cos\theta}}\right) \quad *$$

$$= \frac{32}{3}\left(\frac{\pi}{2} - \int_0^1 1-v^2\,dv\right) = \frac{16\pi}{3} - \frac{32}{3}\left[V - \frac{V^3}{3}\right]_0^1$$

$$= \frac{16\pi}{3} - \frac{32}{3} + \frac{32}{9}$$

$$= \underline{\frac{16\pi}{3} - \frac{64}{9}}$$

$$\underline{\underline{V = \frac{16\pi}{3} - \frac{64}{9}}}$$

Substituering

$u = 4-r^2$
$du = -2r\,dr \Rightarrow dr = \frac{du}{-2r}$
$r = 2\sin\theta \Rightarrow u = 4-4\sin^2\theta$
$\underline{u = 4\cos^2\theta}$
$r = 0 \Rightarrow \underline{u=4}$

Substituering

$V = \sin\theta$
$dv = \cos\theta\,d\theta$
$\Rightarrow d\theta = \frac{dv}{\cos\theta}$
$\theta = \pi/2 \Rightarrow V = 1$
$\theta = 0 \Rightarrow V = 0$

$* \Rightarrow \left[\cos^2\theta = 1-\sin^2\theta = 1-v^2\right]$